"Prediction is very difficult, especially about the future."

N. Bohr

QUANTUM RETRODICTION

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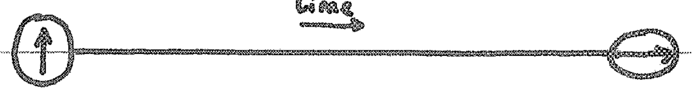
1. Prediction & retrodiction

Consider a spin-$1/2$ particle:

time →



Preparation Measurement

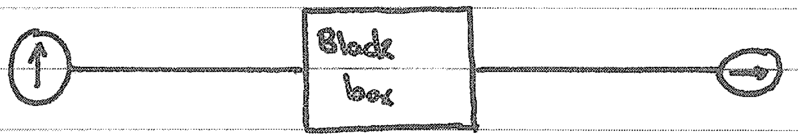
What is the state for times between the preparation and measurement events? How do you know?

Prediction: statements about the future based on presently available knowledge.

Retrodiction (postdiction): statements about the past based on presently available information.

In both cases we cannot usually be certain so we need to rely on probabilistic statements.

time →



Can we predict the results of any measurement in the black box?

Yes, use the state $|\uparrow\rangle$

Can we retrodict the results of any measurements in the black box?

Yes, use the state $|\rightarrow\rangle$.

2. Classical retrodiction: Bayesian inference

Reverend Thomas Bayes (1702 – 1761)

- probabilities depend on what we know
- if we acquire additional information then this modifies the probability.

Conditional probabilities

Consider two events, A and B, with outcomes $\{a_i\}$ & $\{b_j\}$.

Joint probabilities $P(a_i, b_j)$ ← "and"

$$P(a_i) = \sum_j P(a_i, b_j)$$

$$P(b_j) = \sum_i P(a_i, b_j)$$

If we learn that $A = a_0$ then the b_j probabilities change to

$$P(b_j | a_0) \neq P(b_j)$$

"given"

Bayes rule:

$$P(a_i, b_j) = P(b_j | a_i)\, P(a_i)$$
$$P(a_i, b_j) = P(a_i | b_j)\, P(b_j)$$

$$\Rightarrow \quad P(a_i | b_j) = \frac{P(b_j | a_i)\, P(a_i)}{P(b_j)} = \frac{P(b_j | a_i)\, P(a_i)}{\sum_k P(b_j | a_k)\, P(a_k)}$$

If A happens before B, then $P(b_j | a_i)$ is a predictive probability and $P(a_i | b_j)$ is a retrodictive probability.

Exercise:

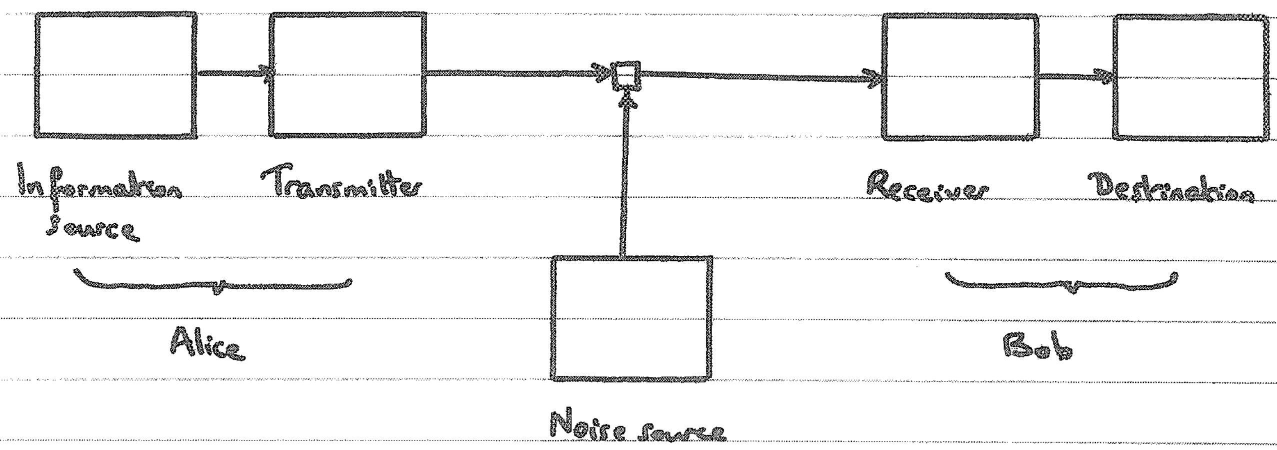
Each day I take a long (ℓ) or short (s) route to work and may arrive on time (O) or late (L). The relevant probabilities are

$$P(b_O | a_s) = 1$$
$$P(b_O | a_\ell) = 3/4$$
$$P(a_\ell) = 1/4 = 1 - P(a_s)$$

Given that you see me arrive on time, what is the probability that I took the long route? (Ans: 1/5).

3. Elements of communications theory

Shannon's model



Alice chooses messages $\{a_i\}$ with prob. $P(a_i)$

Bob receives signals $\{b_j\}$

Alice knows what she sends and so calculates the predictive probabilities $P(b_j|a_i)$.

Bob knows what he receives and so calculates the retrodictive probabilities $P(a_i|b_j)$.

<u>Example</u> (relevant to later discussion):

Suppose our source produces optical pulses (single spatio-temporal mode) with precisely n photons & prob. $P(a_n)$.

Each photon is detected with prob η and missed with prob. $1-\eta$.

$$P(b_N|a_n) = \binom{n}{N}\eta^N(1-\eta)^{n-N}$$

N
counts

Bayes: $$P(a_n|b_N) = \frac{\binom{n}{N}\eta^N(1-\eta)^{n-N}\,P(a_n)}{\sum_{k=N}^{\infty}\binom{k}{N}\eta^N(1-\eta)^{k-N}\,P(a_k)}$$

What if we have "no" prior information? This means all $P(a_n)$ should be equal!?

Try: $P(a_n) = \dfrac{\lambda^n}{1-\lambda}$ and take limit $\lambda \to 1$

$$\Rightarrow \quad P(a_n|b_N) \to \binom{n}{N}\eta^{N+1}(1-\eta)^{n-N}$$

Hence this would be our best guess given no prior information about the number of photons in the pulse.

4. Retrodictive states and dynamics

t_0	t_1
Preparation	Measurement
State $|i\rangle$, one of set $\sum_i |i\rangle\langle i| = \hat{\mathbb{1}}$ with equal prior probs.	Result $|m\rangle$, non-degenerate eigenvalue of measured observable

$$P(f|i) = |\langle f|\hat{U}(t_1,t_0)|i\rangle|^2$$

$$P(i|f) = |\langle i|\hat{U}^{\dagger}(t_1,t_0)|f\rangle|^2$$

(More generally, of course, we need to use Bayes' theorem)

Time-evolution?

$$|i(t)\rangle = \hat{U}(t,t_0)|i\rangle$$

$$i\hbar\frac{d}{dt}|i(t)\rangle = \hat{H}(t)|i(t)\rangle$$

We can also evolve backwards from $|f\rangle$

$$|f(t)\rangle = \hat{U}^{\dagger}(t_1,t)|f\rangle$$

$$\frac{d}{dt}\langle f|\hat{U}(t_1,t_0)|i\rangle = 0 = \frac{d}{dt}\langle f(t)|i(t)\rangle$$

$$= \langle f(t)|\left(\frac{-i}{\hbar}H(t)|i(t)\rangle\right) + \langle \dot{f}(t)|i(t)\rangle$$

$$\Rightarrow \quad \frac{d}{dt}\langle f(t)| = \frac{i}{\hbar}\langle f(t)|\hat{H}(t)$$

$$\text{or} \quad i\hbar\frac{d}{dt}|f(t)\rangle = \hat{H}(t)|f(t)\rangle$$

Same Schrödinger equation but we integrate it with a final boundary condition rather than an initial boundary condition.

5. Bayes' theorem and quantum retrodiction

A more technical section for "specialists". Think again about our communications model but consider quantum communications

Prep^n		Meas.
$\hat{\rho}_i \quad P(a_i)$		$\hat{\pi}_j$ POM (or POVM)

$$\sum_j \hat{\pi}_j = \hat{\mathbb{1}}$$

$$P(b_j|a_i) = \mathrm{Tr}\left(\hat{\rho}_i^{\,pred.}\,\hat{\pi}_j\right)$$

Prior probability for initial state gives $\hat{\rho} = \sum_i P(a_i)\hat{\rho}_i^{\,pred.}$

We say that the preparation is unbiased if

$$\hat{\rho} \propto \hat{\mathbb{1}} \quad \Rightarrow \quad \hat{\rho} = \frac{1}{D}\hat{\mathbb{1}}$$

$D \leftarrow$ dimension of state space.

(for an infinite state-space we need to use a limiting process.)

For an <u>unbiased</u> source we can introduce the preparation analogue of a POM (POVM) with elements

$$\hat{\Xi}_i = D\, P(a_i)\, \hat{\rho}_i^{pred}$$

$$\Rightarrow \sum_i \hat{\Xi}_i = \hat{\mathbb{1}}$$

We can write the retrodictive prob. in the same form as the prediction one:

$$P(b_j | a_i) = Tr\,(\hat{\rho}_i^{pred}\, \hat{\pi}_j)$$

$$P(a_i | b_j) = Tr\,(\hat{\rho}_j^{retr}\, \hat{\Xi}_i)$$

Where $\boxed{\hat{\rho}_j^{retr} = \dfrac{\hat{\pi}_j}{Tr(\hat{\pi}_j)}}$

<u>Proof</u>: (use Bayes' theorem)

$$P(a_i | b_j) = \frac{P(b_j | a_i)\, P(a_i)}{P(b_j)}$$

$$\Rightarrow Tr\,(\hat{\rho}_j^{retr}\, \hat{\Xi}_i) = \frac{Tr\,(\hat{\pi}_j\, \hat{\rho}_i^{pred})\, P(a_i)}{P(b_j)}$$

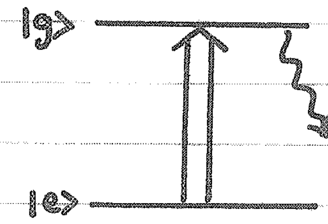
$$= \frac{Tr\,(\hat{\Xi}_i\, \hat{\pi}_j)}{D\, P(b_j)}$$

$$= \frac{Tr\,(\hat{\Xi}_i\, \hat{\pi}_j)}{Tr(\hat{\pi}_j)}$$

6. Retrodiction in quantum optics

We can use retrodictive quantum theory to analyze and reinterpret familiar quantum optical phenomena. Indeed many of our physical "explanations" for these are intrinsically retrodictive in nature.

6.1 Photon anti-bunching



Consider a single two-level atom (or trapped ion) driven by a laser. The fluorescence (or rather a small part of it) is collected on a photodiode.

If a photon is detected at time t_1 we have to wait a significant time, Δt, before there is a reasonable probability for detecting another.

Retrodictive explanation:

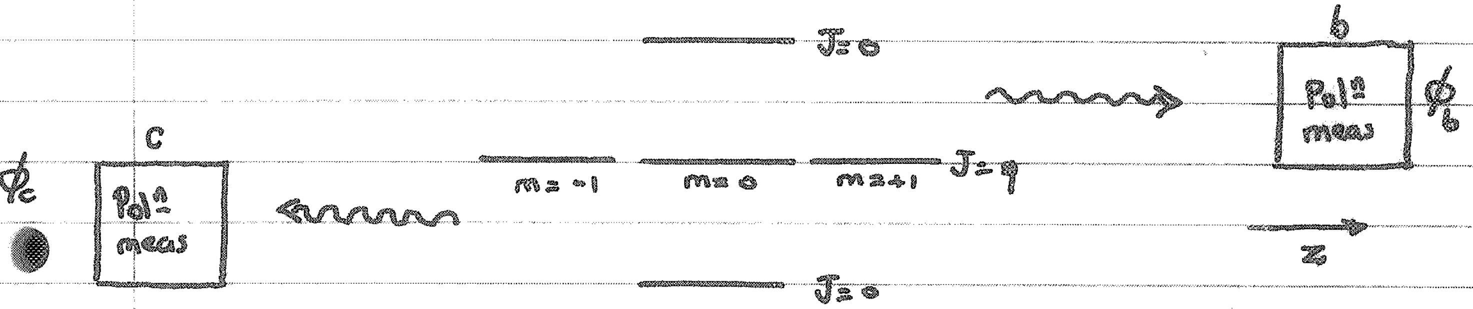
A detection at time t_1 corresponds, retrodictively, to a fluorescent photon emitted at time $t_1 - d/c$. Hence the retrodictive state of the atom is

$$|\psi(t_1 - d/c)\rangle^{retr} = |g\rangle$$

We can now work forwards in time, or predictively, using this "collapse". It takes a time Δt ($\sim 1/\Omega_{Rabi}$) before the atom has a significant probability to be in the excited state $|e\rangle$, which allows it to fluoresce. Hence there is a delay $\gtrsim \Delta t$ between photocounts.

6.2 Kocher - Commins experiment

Consider a two-photon cascade emission from a $J=0$ excited state, through a $J=1$ state, to a $J=0$ ground state.



The transition occurs via the $m=\pm1$ intermediate states.

Predictive formalism:

The two photons are emitted in the pol^n entangled state

$$\frac{1}{\sqrt{2}}\left(|x_b\rangle|x_c\rangle + |y_b\rangle|y_c\rangle\right)$$

Detection at b projects this state onto $\cos\phi_b|x_b\rangle + \sin\phi_b|y_b\rangle$ which then collapses the distant photon onto the state $\cos\phi_b|x_c\rangle + \sin\phi_b|y_c\rangle$. The probability amplitude for detecting both photons is then

$$\cos\phi_b\cos\phi_c + \sin\phi_b\sin\phi_c = \cos(\phi_b - \phi_c)$$

as it should be.

Retrodictive formalism:

Detecting a photon with polarization $\cos\phi_b |x_b\rangle + \sin\phi_b |y_b\rangle$ means that a photon with this polarization was emitted from the excited $J=0$ state.

$$\cos\phi_b |x_b\rangle + \sin\phi_b |y_b\rangle = e^{i\phi_b} \frac{1}{2}(|x_b\rangle - i|y_b\rangle) + e^{-i\phi_b} \frac{1}{2}(|x_b\rangle + i|y_b\rangle)$$

Hence the intermediate ($J=1$) state of the atom was

$$\frac{1}{\sqrt{2}}\left(-e^{-i\phi_b}|m=-1\rangle + e^{i\phi_b}|m=+1\rangle\right)$$

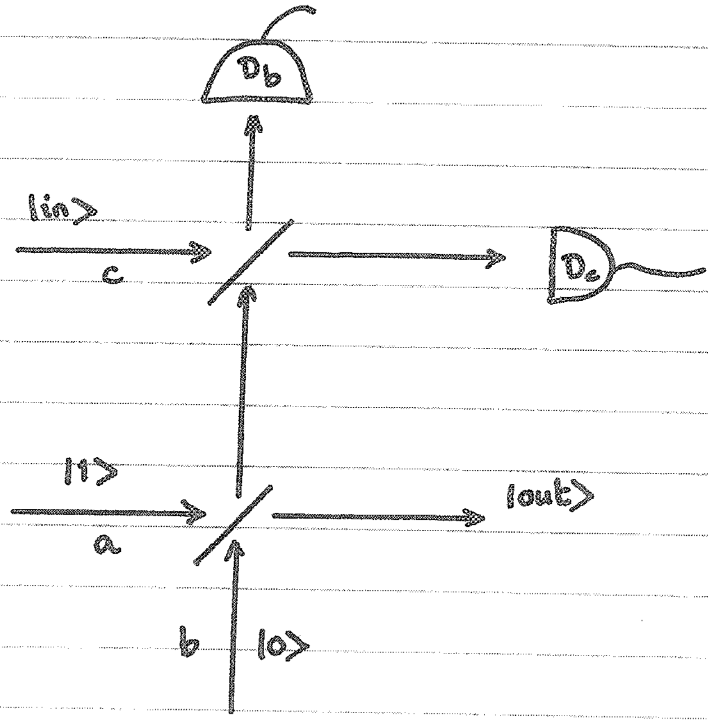
This then radiates the second photon with a well-defined polarization state:

$$\cos\phi_b |x_c\rangle + \sin\phi_b |y_c\rangle$$

The detection probability for this second photon then follows the simple Malus law.

6.3 Quantum scissors

We can use quantum retrodiction as a tool to analyze measurement devices of some subtlety that would be more difficult to understand purely predictively.



Let $|in\rangle$ be the superposition state

$$|in\rangle = c_0|0\rangle_c + c_1|1\rangle_c + c_2|2\rangle_c + \dots$$

e.g. $$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle_c$$

Consider what happens when detector D_b registers one photon and detector D_c registers no photons.

Predictive analysis:

Sketch analysis. The steps are (i) work out the entangled output state produced by the lower beam-splitter (ii) interfer the output mode b with mode c on the second beam-splitter (iii) finally, project the three-mode entangled state onto the state $|1\rangle_b|0\rangle_c$ to get the state $|out\rangle$. Phew!

Retrodictive analysis:

We *start* with the detected state $|1\rangle_b |0\rangle_c$ and propagate this back to the upper beam-splitter. If we interfer this state at the upper beam-splitter then we get the retrodictive state before the beam-splitter

$$\frac{1}{\sqrt{2}}\left(|1\rangle_b |0\rangle_c - i\,|0\rangle_b |1\rangle_c\right)$$

Project this onto the state $|in\rangle$ and we get

$$\langle in|\left(|1\rangle_b |0\rangle_c - i\,|0\rangle_b |1\rangle_c\right) = i c_0^* |1\rangle_b + c_1^* |0\rangle_b$$

(ignoring annoying normalization factors).

We can now project this onto the output state,

$$\frac{1}{\sqrt{2}}\left(|1\rangle_a |0\rangle_b + i|0\rangle_a |1\rangle_b\right),$$

of the lower beam-splitter

$$|out\rangle = \left(-i c_0\, {}_b\langle 1| + c_1\, {}_b\langle 0|\right)\left(|1\rangle_a |0\rangle_b + i\,|0\rangle_a |1\rangle_b\right)$$

$$= c_0 |0\rangle_a + c_1 |1\rangle_a$$

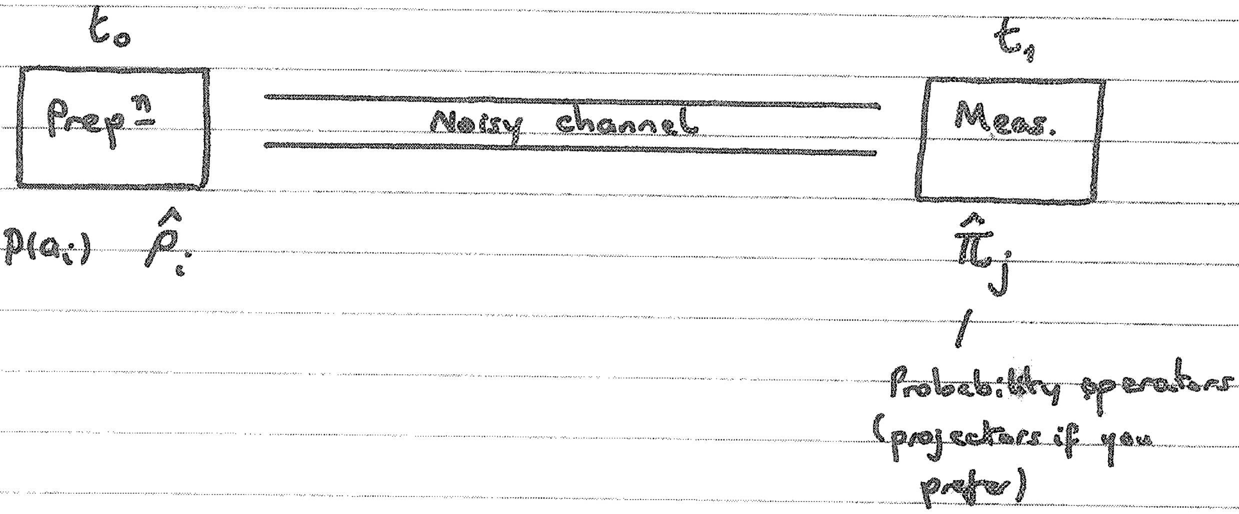
Hence we have a truncated form of the input state

$$c_0 |0\rangle + c_1 |1\rangle + c_2 |2\rangle + \ldots$$

Copenhagen interpretation? Where is the collapse? Ans. here we have it between the beam-splitters. More generally, you can put it anywhere you like!

7. Open - system dynamics

Consider again our quantum communication channel



$$\dot{\hat{\rho}}_i = -\frac{i}{\hbar}[\hat{H},\hat{\rho}_i] + \gamma_k \sum_k 2\hat{A}_k \hat{\rho}_i \hat{A}_k^{+} - \hat{A}_k^{+}\hat{A}_k\hat{\rho}_i - \hat{\rho}_i \hat{A}_k^{+}\hat{A}_k$$

Lindblad form. Exercise: confirm that $\mathrm{Tr}(\dot{\hat{\rho}}_i) = 0$.

$$P(b_j|a_i) = \mathrm{Tr}(\hat{\rho}_i(t_1)\hat{\pi}_j)$$

We can think of evolving $\hat{\rho}_i$ to time t <u>and</u> $\hat{\pi}_i$ back to time t $(t_0 < t < t_1)$

$$\frac{d}{dt}P(b_j|a_i) = 0$$

$$= \mathrm{Tr}[\dot{\hat{\rho}}_i(t)\hat{\pi}_j(t)] + \mathrm{Tr}[\hat{\rho}_i(t)\dot{\hat{\pi}}_j(t)]$$

$$\Rightarrow \quad -\frac{i}{\hbar}\,\mathrm{Tr}\left(\left[\hat{H},\hat{\rho}_i\right]\hat{\pi}_j\right) + \sum_k \gamma_k\,\mathrm{Tr}\left(2\hat{A}_k\hat{\rho}_i\hat{A}_k^\dagger\hat{\pi}_j - \hat{A}_k^\dagger\hat{A}_k\hat{\rho}_i\hat{\pi}_j - \hat{\rho}_i\hat{A}_k^\dagger\hat{A}_k\hat{\pi}_j\right)$$

$$= -\,\mathrm{Tr}\left(\hat{\rho}_i\dot{\hat{\pi}}_j\right)$$

$$\Rightarrow \quad \dot{\hat{\pi}}_j = -i\left[\hat{H},\hat{\pi}_j\right] - \sum_k \gamma_k 2\,\underbrace{\hat{A}_k^\dagger\hat{\pi}_j\hat{A}_k}_{\text{note order.}}$$

$$+ \sum_k \gamma_k\left[\hat{A}_k^\dagger\hat{A}_k\hat{\pi}_j + \hat{\pi}_j\hat{A}_k^\dagger\hat{A}_k\right]$$

Not of Lindblad form <u>BUT</u> preserves the physically important property $\sum_j \hat{\pi}_j = \hat{\mathbb{1}}$. (Exercise to prove this).

Simple example: consider a single optical mode ~~mode~~ undergoing linear loss

$$\dot{\hat{\rho}}_i = 2\gamma\hat{a}\hat{\rho}_i\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho}_i - \hat{\rho}_i\hat{a}^\dagger\hat{a}$$

What do we need to do to the retrodictive state $\hat{\rho}_j^{retr}$, i.e. given that we measure $\hat{\pi}_j$, what state should we infer at the input?

$$\dot{\hat{\pi}}_j = -2\gamma\hat{a}^\dagger\hat{\pi}_j\hat{a} + \hat{a}^\dagger\hat{a}\hat{\pi}_j + \hat{\pi}_j\hat{a}^\dagger\hat{a}$$

Recall that

$$\hat{\rho}_j^{retr} = \frac{\hat{\pi}_j}{\mathrm{Tr}(\hat{\pi}_j)}$$

$$\mathrm{Tr}\,\dot{\hat{\pi}}_j = \gamma\,\mathrm{Tr}\left(-2\hat{a}^+\hat{\pi}_j\hat{a} + \hat{a}^+\hat{a}\,\hat{\pi}_j + \hat{\pi}_j\hat{a}^+\hat{a}\right)$$

$$= -2\gamma\,\mathrm{Tr}\left(\hat{\pi}_j[\hat{a},\hat{a}^+]\right)$$

$$= -2\gamma$$

$$\therefore \mathrm{Tr}\,\hat{\pi}_j = \mathrm{Tr}(\hat{\pi}_j(t_1))\,e^{2\gamma(t_1-t)}$$

(recall that we integrate backwards in time!)

$$\Rightarrow \dot{\hat{\rho}}_j^{retr} = -\gamma\left(2\hat{a}^+\hat{\rho}_j^{retr}\hat{a} - \hat{a}\hat{a}^+\hat{\rho}_j^{retr} - \hat{\rho}_j^{retr}\hat{a}\hat{a}^+\right)$$

Naturally enough, perhaps, retrodiction through a channel with linear losses, corresponds to forward evolution through an amplifying channel.

If we want to retrodict, for example, an n-photon detection, then we find the retrodicted state

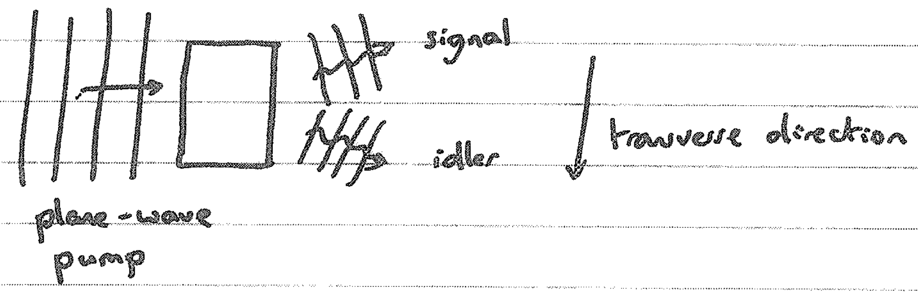
$$\hat{\rho}_n^{retr} = \sum_{N=n}^{\infty}\binom{n}{N}e^{-\gamma(t_1-t_0)(N+1)}\left[1-e^{-\gamma(t_1-t_0)}\right]^{n-N}|N\rangle\langle N|$$

If we write $\eta = e^{-\gamma(t_1-t_0)}$ then we get, naturally enough, the result we found in section 3 for finite efficiency photodetection.

8. Case study: spontaneous parametric down-conversion

Spontaneous parametric down-conversion is a source of entangled photon pairs. Momentum conservation requires that the sum of the momenta for the two photons matches the momentum of a photon in the laser pump.

It suffices to consider the momentum k in the transverse direction and consider just one transverse direction.



We can write our two-photon state as a "biphoton wavefunction"

$$\psi(k_s, k_i) = N e^{-(k_s + k_i)^2/\Delta^2} e^{-(k_s - k_i)^2/\sigma^2}$$

small

large (determines bandwidth)

$$\sim \delta(k_s + k_i)$$

"Perfect" anticorrelation in transverse momentum.

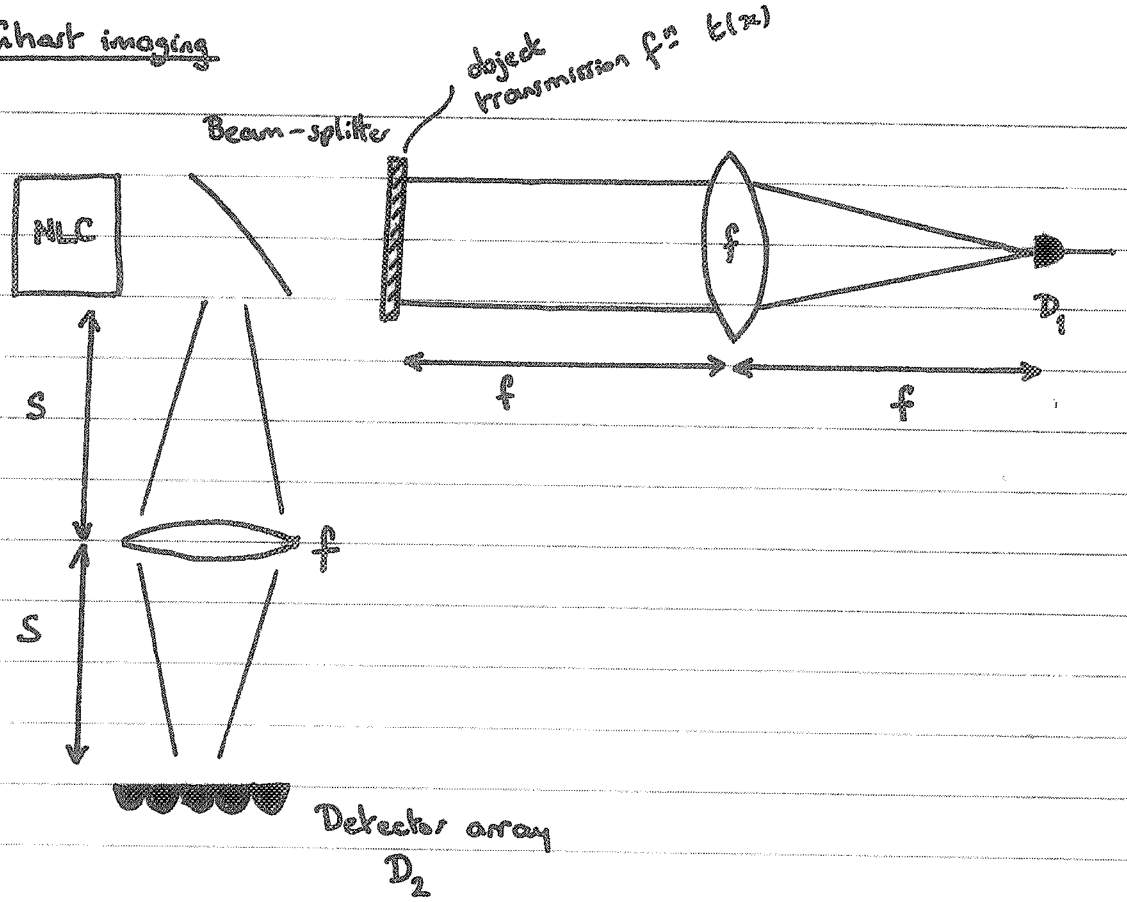
In the position representation we have

$$\tilde{\psi}(x_s, x_i) = \tilde{N} e^{-\Delta^2(x_s + x_i)^2/4} e^{-\sigma^2/4 (x_s - x_i)^2}$$

$$\sim \delta(x_s - x_i) \qquad \text{EPR correlations!}$$

Photon pairs are born in the same place.

8.1 Ghost imaging



Measure "clicks" in detector array D_2 in coincidence with detection in D_1.

1. If $s = f$ then the probability distribution in arm 2 is the squared modulus of the Fourier transform of $t(x)$.

2. If $s = 2f$ then the probability distribution in arm 2 is $|t(x)|^2$.

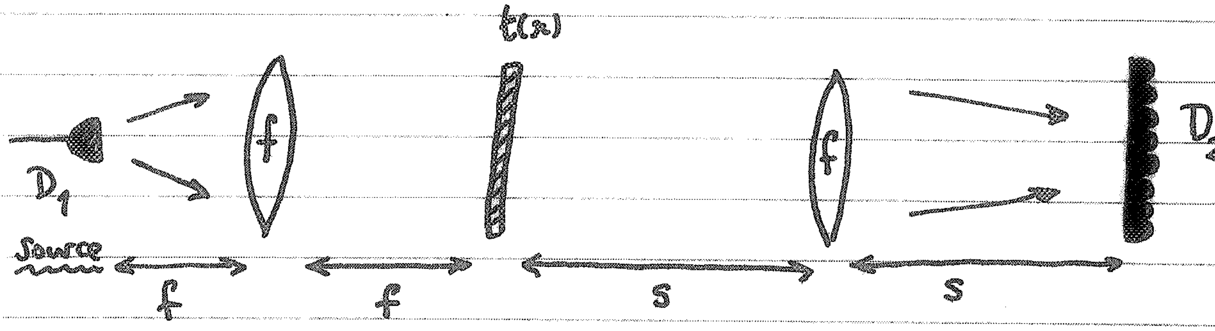
The light forming the image never passed through the object hence "ghost imaging"

NB the effect can be seen also (to some extent) with "classical" chaotic light.

What's going on?

We can derive these results us our biphoton wavefunction. A very simple way to proceed, however, is to use the Klyshko interpretation. The idea is simply to start at detector D_1 and consider this as a source of an "advanced potential" that propagates back in time to the crystal. If we replace the crystal by a "mirror in time" we get the observed images.

To think in one direction in time, replace the SPDC system by



For $s = f$ we find the squared modulus of the Fourier transform of $t(x)$

For $s = 2f$ we find $|t(x)|^2$

8.2 Retrodictive description: derivation of the Klyshko interpretation

It is natural to picture the Klyshko interpretation using quantum retrodiction. Indeed we can derive it! Lets sketch this.

We start at the detector D_1 which corresponds to a tightly focussed (approximately Gaussian) spot. The associated mode amplitude is

$$\psi(x_s) \simeq N e^{-x_s^2/w^2}$$

where w is "small". This corresponds, in the retrodictive picture, to a photon source. The photon propagates back through the lens to the object plane, where it has the form

$$\psi^{retr}_{\text{object plane}} = N' e^{-w^2 x_s^2/4}$$

which is approximately a plane wave (very broad). Propagating this back through the object gives

$$t^*(x_s)\, e^{-w^2 x_s^2/4}$$

If we take the overlap of this with the two-photon wavefunction, we have

$$\int dx_s \; t(x_s)\, e^{-w^2 x_s^2/4} \; \delta(x_s - x_i)$$

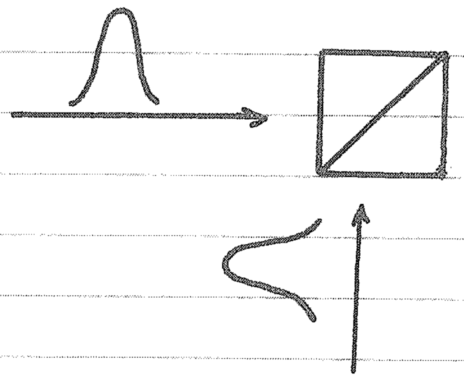
$$= \quad t(x_i)\, e^{-w^2 x_i^2/4}$$

$$\simeq \quad t(x_i)$$

If we propagate this forward towards the detector array then we get either $|t(x_i)|^2$ $(s = 2f)$ or the Fourier transform $(s = f)$.

9. Two-particle interference

Let us consider two single-particle pulses impinging from different directions on a 50/50 beam-splitter. We shall assume that these overlap perfectly in space and in time. The particles are identical.

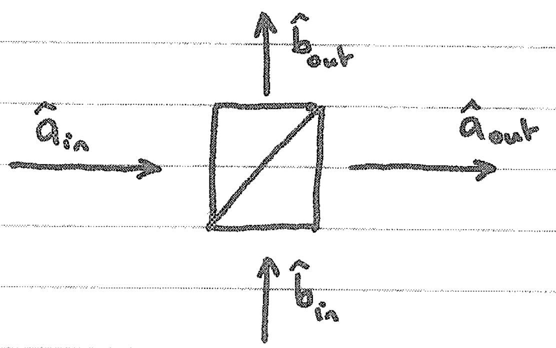


What happens at the output and why?

N.B. photon experiments are usually done with entangled pairs but entanglement is not necessary for this effect.

9.1 Symmetric beam splitter

We can derive relations between the annihilation operators as follows



$$\hat{a}_{out} = t\hat{a}_{in} + r\hat{b}_{in}$$
$$\hat{b}_{out} = t\hat{b}_{in} + r\hat{a}_{in}$$

For bosons we have

$$[\hat{a}_{out}, \hat{a}^{\dagger}_{out}] = 1 = |t|^2 + |r|^2$$ (similarly for $[b_{out}, b^{\dagger}_{out}]$)

$$[\hat{a}_{out}, \hat{b}^{\dagger}_{out}] = 0 = tr^* + rt^*$$ (similarly for $[b_{out}, a^{\dagger}_{out}]$)

Exercise: show that the same relations hold for fermions.

A simple choice is to make t real and r imaginary.

9.2 Two input particles

Predictive analysis:

The input state is

$$\hat{a}^{\dagger}_{in}\hat{b}^{\dagger}_{in}|0\rangle = (t\hat{a}^{\dagger}_{out} + r\hat{b}^{\dagger}_{out})(t\hat{b}^{\dagger}_{out} + r\hat{a}^{\dagger}_{out})|0\rangle$$

$$= \Big[t^2\hat{a}^{\dagger}_{out}b^{\dagger}_{out} + r^2 b^{\dagger}_{out}a^{\dagger}_{out} + tr\left(\hat{a}^{\dagger 2}_{out} + \hat{b}^{\dagger 2}_{out}\right)\Big]|0\rangle$$

1. Bosons

$$\hat{a}^{\dagger}_{out}\hat{b}^{\dagger}_{out} = \hat{b}^{\dagger}_{out}\hat{a}^{\dagger}_{out}$$

$$t^2 = |t|^2, \quad r^2 = -|r|^2$$

$$\therefore \hat{a}^{\dagger}_{in}\hat{b}^{\dagger}_{in}|0\rangle = \Big[\underbrace{(|t|^2 - |r|^2)}_{\text{two-photon interference}}\hat{a}^{\dagger}_{out}\hat{b}^{\dagger}_{out} + tr(\hat{a}^{\dagger 2}_{out} + \hat{b}^{\dagger 2}_{out})\Big]|0\rangle$$

For $|t| = |r|$, the two photons always leave in the same direction.

2. Fermions

$$\hat{a}^{\dagger}_{out}\hat{b}^{\dagger}_{out} = -\hat{b}^{\dagger}_{out}\hat{a}^{\dagger}_{out}$$

$$\therefore \quad \hat{a}^{\dagger}_{in}\hat{b}^{\dagger}_{in}|0\rangle = \Big[\underbrace{(|t|^2+|r|^2)}_{=1}\hat{a}^{\dagger}_{out}\hat{b}^{\dagger}_{out} + tr\big(\cancel{\hat{a}^{\dagger 2}_{out}} + \cancel{\hat{b}^{\dagger 2}_{out}}\big)\Big]|0\rangle$$

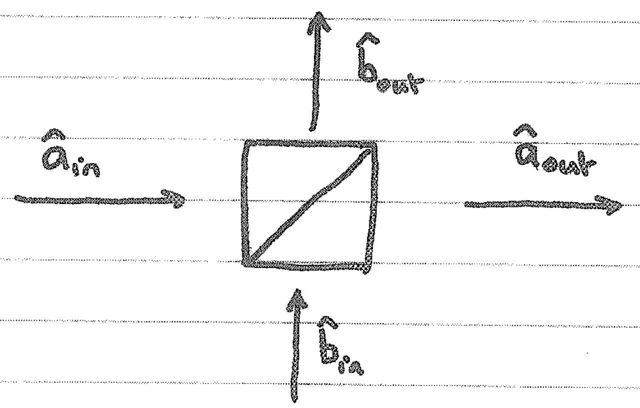
$$= \hat{a}^{\dagger}_{out}\hat{b}^{\dagger}_{out}|0\rangle$$

as required by Pauli's exclusion principle.

9.8 Retrodictive explanation

The Klyshko interpretation of ghost imaging involves just one photon. It propagates from detector D_1 back in time to the nonlinear crystal and then forward to the detector array D_2.

Qu: does quantum retrodiction provide a single-particle explanation of two-particle interference?



To answer this let us suppose that we detect one particle in output mode $\hat{a}_{out}$ and try to use this to infer where the other particle emerges.

Our single-particle retrodictive state is

$$\hat{a}^{\dagger}_{out}|0\rangle$$

If we propagate this back through the beam-splitter then we find the retrodictive state

$$\frac{1}{\sqrt{2}}(\hat{a}^{\dagger}_{in} - i\hat{b}^{\dagger}_{in})|0\rangle.$$

Taking the overlap of this with our two-particle input state gives

$$\frac{1}{\sqrt{2}} \quad (\hat{a}_{in} + i\hat{b}_{in})\,\hat{a}^{\dagger}_{in}\hat{b}^{\dagger}_{in}|0\rangle$$

1. Bosons

The resulting state of the other particle is

$$\frac{1}{\sqrt{2}}(\hat{b}^{\dagger}_{in} + i\hat{a}^{\dagger}_{in})|0\rangle \qquad \begin{pmatrix} \hat{a}\hat{a}^{\dagger}\hat{b}^{\dagger}|0\rangle = \hat{b}^{\dagger}|0\rangle \\ \hat{b}\hat{a}^{\dagger}\hat{b}^{\dagger}|0\rangle = \hat{a}^{\dagger}|0\rangle \end{pmatrix}$$

If we propagate this forward through the beam-splitter we find

$$\frac{1}{\sqrt{2}}(\hat{b}^{\dagger}_{in} + i\hat{a}^{\dagger}_{in})|0\rangle \longrightarrow \frac{1}{2}\left[\hat{b}^{\dagger}_{out} + i\hat{a}^{\dagger}_{out} + i(\hat{a}^{\dagger}_{out} + i\hat{b}^{\dagger}_{out})\right]|0\rangle$$

$$= i\hat{a}^{\dagger}_{out}|0\rangle$$

so the retrodictive analysis gives the result that the "second" particle emerges in the same direction as the first.

2. Fermions

Now we need to be careful! Our overlap with the two-particle input state gives

$$\frac{1}{\sqrt{2}}(\hat{a}_{in} + i\hat{b}_{in})\,\hat{a}^{+}_{in}\hat{b}^{+}_{in}|0\rangle$$

$$= \left(\frac{1}{\sqrt{2}}\hat{a}_{in}\hat{a}^{+}_{in}\hat{b}^{+}_{in} + \frac{i}{\sqrt{2}}\hat{b}_{in}\hat{a}^{+}_{in}\hat{b}^{+}_{in}\right)|0\rangle$$

anticommutation! $= \frac{1}{\sqrt{2}}\left(\hat{b}^{+}_{in}\hat{a}_{in}\hat{a}^{+}_{in} - i\hat{a}^{+}_{in}\hat{b}_{in}\hat{b}^{+}_{in}\right)|0\rangle$

$$= \frac{1}{\sqrt{2}}\left(\hat{b}^{+}_{in} - i\hat{a}^{+}_{in}\right)|0\rangle$$

sign-change due to fermionic nature.

If we propagate this forward through the beam-splitter we find

$$\frac{1}{\sqrt{2}}\left(\hat{b}^{+}_{in} - i\hat{a}^{+}_{in}\right)|0\rangle = \frac{1}{2}\left[\hat{b}^{+}_{out} + i\hat{a}^{+}_{out} - i(\hat{a}^{+}_{out} + i\hat{b}^{+}_{out})\right]|0\rangle$$

$$= \hat{b}^{+}_{out}|0\rangle$$

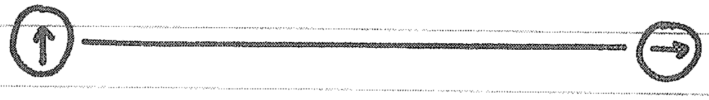
So the retrodictive analysis shows that the "second" fermionic particle emerges in the <u>other</u> output to the first.

10. Retrodiction and the "quantum arrow of time"

For many physicists, there is a connection between quantum measurement and/or state collapse and the arrow of time. Those who favour the many-worlds interpretation see a universe branching into ever more universes as we move into the future.

To some extent these ideas presuppose the existence of a state, perhaps a physical existence. Measurement, for example, may be viewed as assisted by some extra new physics, e.g. a gravitational interaction.

Retrodiction is more Bayesian in nature. The state depends on what we know. In our spin-½ example



Alice Bob

Alice would write the state $|\uparrow\rangle$ between prep^{n} and measurement, but Bob would write $|\rightarrow\rangle$.

Using retrodictive reasoning, moreover, we can delocalise any "collapse". We can think of it happening at any time between preparation and measurement.

The many-worlds interpretation is similarly a bit unnatural. This is because, in retrodictive theory, the new worlds appear as we propagate into the past!

11. <u>A few references</u> (more in the notes to be published)

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