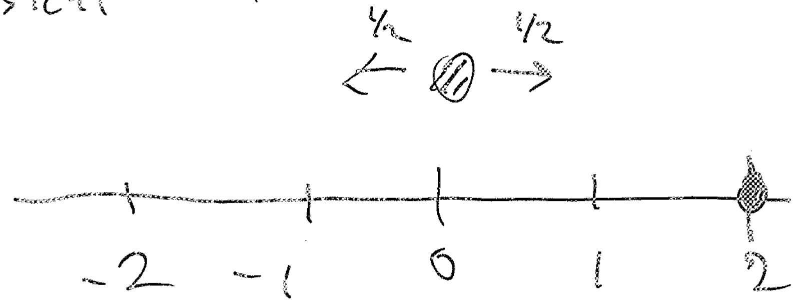
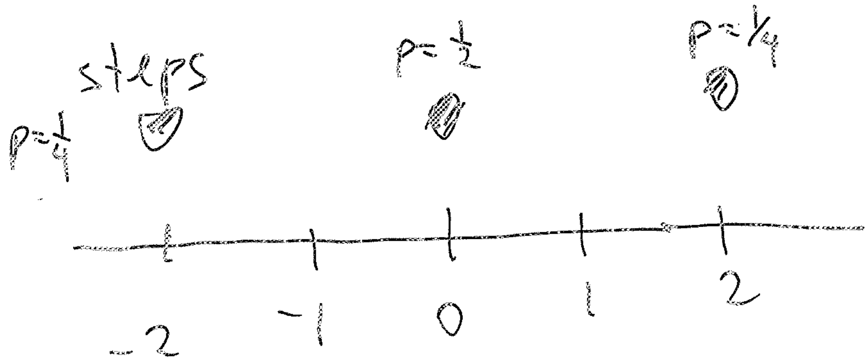
Quantum Walks

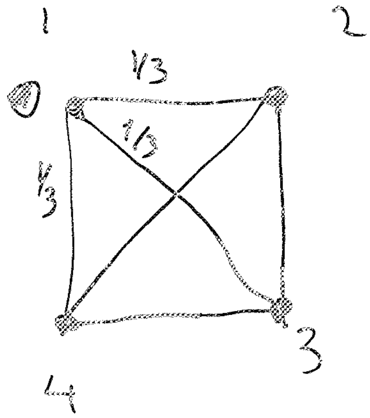
Classical random walk



two steps

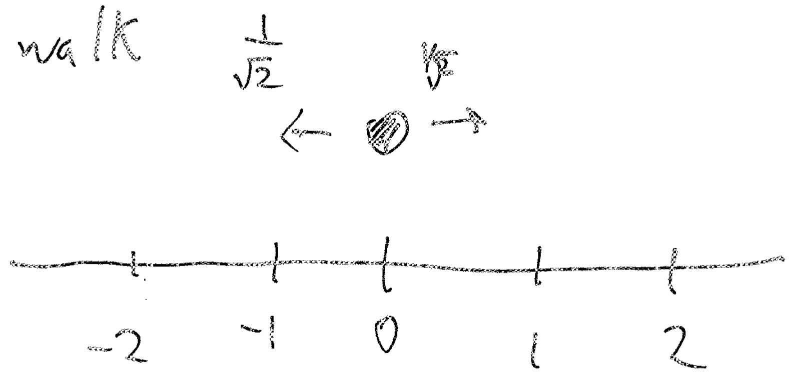


walk on graph



Quantum walk

First try



$|-1\rangle$ $|0\rangle$ $|1\rangle$ $|2\rangle$ ← orthonormal basis

$|0\rangle \to \frac{1}{\sqrt{2}}(|-1\rangle + |1\rangle)$

$|2\rangle \to \frac{1}{\sqrt{2}}(|1\rangle + |3\rangle)$

↑ orthogonal ↑ not orthogonal

but operator must be unitary

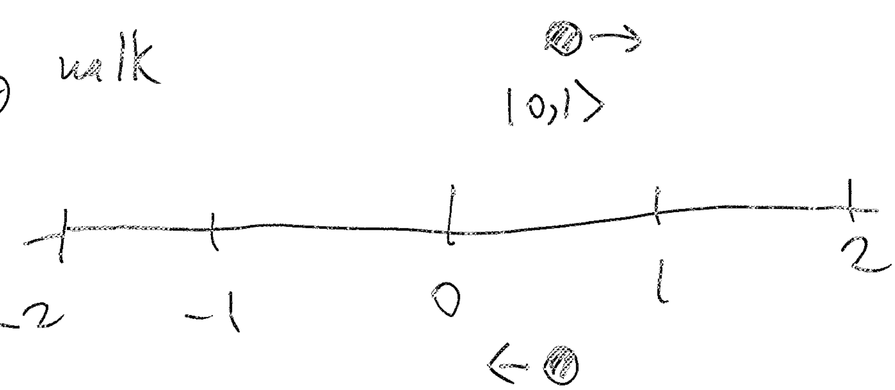
Won't work

what to do

① add a coin
② scattering walk } discrete time steps
③ continuous-time walk

Scattering walk

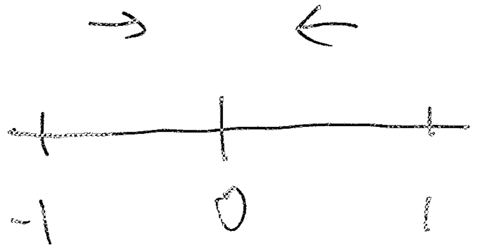
$|0,1\rangle$



$|1,0\rangle$

particle is on edges

Collection of all these basis for Hilbert space



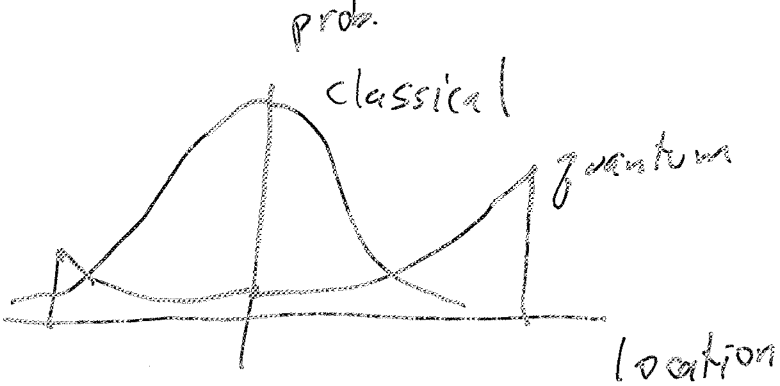
beam splitter at each vertex

$$|-1,0\rangle \to \frac{1}{\sqrt{2}}\left(|0,1\rangle + |0,-1\rangle\right)$$

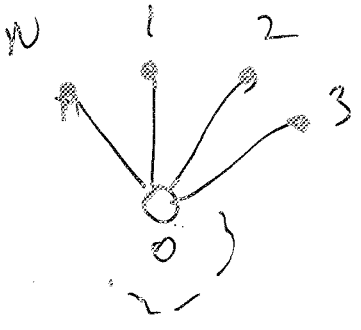
$$|1,0\rangle \to \frac{1}{\sqrt{2}}\left(|0,-1\rangle - |0,1\rangle\right)$$

each vertex acts like this

Start in $|0,1\rangle$ n steps



Use this to do a search



star graph

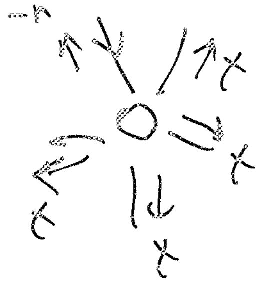
One outside vertex special; want to find it.

$U =$ operator that advances walk 1 step

$$U|0,j\rangle = \begin{cases} -|1,0\rangle & j=1 \\ |j,0\rangle & j>1 \end{cases}$$

1 is special vertex

Central vertex



$$U|j,0\rangle = -r|0,j\rangle + t\sum_{k\neq j}|0,k\rangle$$

① unitary

② each of entering edges behaves in same way

$$r = \frac{N-2}{N} \qquad t = \frac{2}{N}$$

Initial state

$$|\psi_{init}\rangle = \frac{1}{\sqrt{N}}\sum_{j=1}^{N}|0,j\rangle$$

Dimension of space $= 2N$

Walk takes place in a 4D subspace

$$|\psi_1\rangle = |0,1\rangle$$

$$|\psi_2\rangle = |1,0\rangle$$

$$|\psi_3\rangle = \frac{1}{\sqrt{N-1}}\sum_{j=2}^{N}|0,j\rangle$$

$$|\psi_4\rangle = \frac{1}{\sqrt{N-1}}\sum_{j=2}^{N}|j,0\rangle$$

$S = \mathrm{span}\{|\psi_1\rangle \ldots |\psi_4\rangle\}$

$$U: S \rightarrow S$$

$|\psi_{init}\rangle = \frac{1}{\sqrt{N}}|\psi_1\rangle + \sqrt{\frac{N-1}{N}}\,|\psi_3\rangle \in S$

walk starts in S and stays in S

$$U \text{ confined to } S = \begin{pmatrix} 0 & -r & 0 & t\sqrt{N-1} \\ -1 & 0 & 0 & 0 \\ 0 & t\sqrt{N-1} & 0 & r \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Need $U^m|\psi_{init}\rangle$

Diagonalize

$|u_1\rangle \leftrightarrow e^{i\Delta}$

$|u_2\rangle \leftrightarrow e^{-i\Delta}$

$|u_3\rangle \leftrightarrow -e^{-i\Delta}$

$|u_4\rangle \leftrightarrow -e^{i\Delta}$

$\Delta = \sqrt{\frac{t}{2}} = \frac{1}{\sqrt{N}}$

$$U^m = e^{im\Delta}|u_1\rangle\langle u_1| + e^{-im\Delta}|u_2\rangle\langle u_2| + (-1)^m e^{-im\Delta}|u_3\rangle\langle u_3| + (-1)^m e^{im\Delta}|u_4\rangle\langle u_4|$$

Apply to initial state

$$U^m|\psi_{initial}\rangle = \frac{1}{2}\begin{pmatrix} \sin(m\Delta) \\ -\sin(m\Delta) \\ \cos(m\Delta) \\ \cos(m\Delta) \end{pmatrix} + \frac{1}{2}(-1)^m \begin{pmatrix} \sin(m\Delta) \\ \sin(m\Delta) \\ \cos(m\Delta) \\ -\cos(m\Delta) \end{pmatrix}$$

original basis

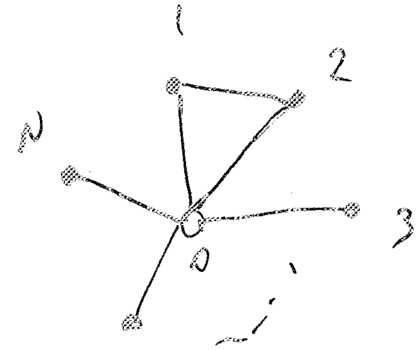
Choose $m\Delta \cong \frac{\pi}{2}$ (m integer)

m even in state $|0,1\rangle$ } on edge connected to
m odd in state $-|1,0\rangle$ } special vertex

Measure position of particle $\rightarrow$ special vertex

How many steps
$$m \cong \frac{\pi}{2\Delta} = \frac{\pi}{2}\sqrt{N}$$

Quantum walk version of Grover



Add extra edge between two external vertices

Can be between any two vertices.

Can we find the extra edge?

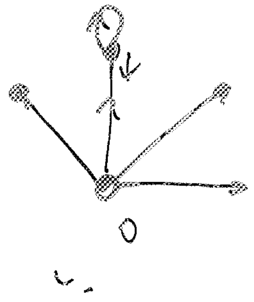
$U|0,j\rangle = |j,0\rangle \quad j > 2$

$U|0,1\rangle = |1,2\rangle \qquad U|0,2\rangle = |2,1\rangle$

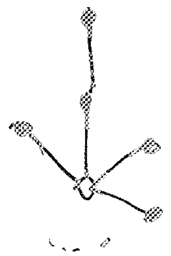
$U|1,2\rangle = |2,0\rangle \qquad U|2,1\rangle = |1,0\rangle$

In this case walk takes place in 5 D subspace.

what kinds of structural anomalies can one find?



works



does not work

Open questions: what works and what doesn't and why?