What is information

Represent content in binaries. 0
1

Communication

How can we make transmission most efficient?

Extreme example I:

Digital camera picture of white sheet of paper. ⇝ Represent content of the picture in a very compact form.

"All pixels are white"

Shannon 1948 or so, ⇝ Information theory

For now: Each entry of the memory is independent of the next.

0 1 0 1 1 1

P_0

P_1

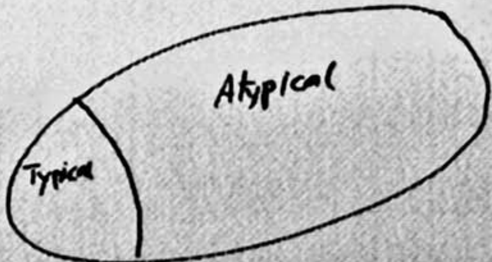
How much can I shorten the representation of the message without error?

$p_0 = \frac{1}{8}$

$p_1 = \frac{7}{8}$

Enumerate

0 1 1 1 1 1 1 1	0 0 0
1 0 1 1 1 1 1 1	0 0 1
1 1 0 1 1 1 1 1	0 1 0
1 1 1 0 1 1 1 1	0 1 1
1 1 1 1 0 1 1 1	1 0 0
1 1 1 1 1 0 1 1	1 0 1
1 1 1 1 1 1 0 1	1 1 0
1 1 1 1 1 1 1 0	1 1 1



Length of sequences $\to \infty$ } Law of large numbers.
$p(\text{Typical sequence}) \to 1$

Given N digits: How many typical seq.?

$$\binom{N}{Np_0} = \frac{N!}{(Np_0)!\,(N(1-p_0))!} = \frac{N^N e^{-N}\sqrt{2\pi N}}{(Np_0)^{Np_0} e^{-Np_0}\sqrt{2\pi N p_0}\,(Np_1)^{Np_1} e^{-Np_1}\sqrt{2\pi N p_1}}$$

$K! \cong K^K e^{-K}\sqrt{2\pi K}$
Stirlings formula

$$= \frac{1}{\sqrt{2\pi N p_0 p_1}} \cdot \frac{N^N}{N^{Np_0} p_0^{Np_0}\, N^{Np_1} p_1^{Np_1}}$$

$$= \frac{1}{\sqrt{2\pi N p_0 p_1}} \cdot \left(\frac{1}{p_0^{p_0}\, p_1^{p_1}}\right)^N$$

$$\log_2 \binom{N}{Np_1} \cong -\frac{1}{2}\log_2(2\pi N p_0 p_1) - N \log_2 p_0^{p_0} p_1^{p_1}$$

$$= \quad -\|- \quad - N(p_0 \log p_0 + p_1 \log p_1)$$

$$\cong N\, H(\{p_0\})$$

$$H(\{p_0\}) = -p_0 \log p_0 - p_1 \log p_1$$

Shannon entropy

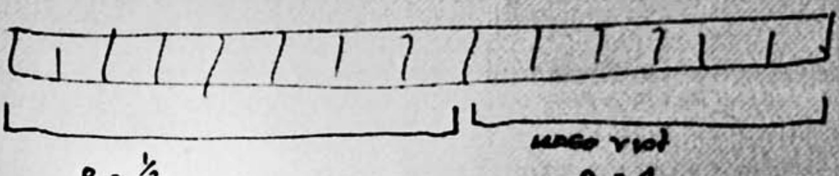
Can compress the original message by a factor $H(p_0)$.

<u>Check:</u> $p_0 = 0$, $p_1 = 1$

$$H(p_0, p_1) = -0 \cdot \log 0 - 1 \log 1 = 0 - 0 = 0$$

$p_0 = p_1 = 1/2$

$$H(p_0) = -\tfrac{1}{2}\log\tfrac{1}{2} - \tfrac{1}{2}\log\tfrac{1}{2} = -\log\tfrac{1}{2} = \log 2 = 1$$



P_0 $\quad p_{00} = 1$

$p_{01} = 0 = p_{10} = p_{11}$

$$\begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \vdots & \end{matrix}$$

uncorrelated

$I = 0$

<u>Cryptography</u>

Secret info		Secret correlation			
0	$\oplus$	$0 = 0$	$\oplus$	0	$= 0$
1	$\oplus$	$1 = 0$	$\oplus$	1	$= 1$
1	$\oplus$	$1 = 0$	$\oplus$	1	$= 1$
1	$\oplus$	$1 = 0$	$\oplus$	1	$= 1$
0	$\oplus$	$0 = 0$	$\oplus$	0	$= 0$
1	$\oplus$	$0 = 1$	$\oplus$	0	$= 1$
0	$\oplus$	$1 = 1$	$\oplus$	1	$= 0$
1					

public communication

Secret correlations are a resource for secret public communication.

"One time pad"

<u>Exercise:</u> Transmit digit a via first entry of your "one time pad"

— " — b — " —

Can anybody ~~l~~ learn something about a & b?

Correlations

~~$\langle \sigma_x^{(1)} \sigma_x^{(2)} \rangle$~~

Ex.:

$p_{00} = \frac{1}{2}$

$p_{01} = p_{10} = 0$

$p_{11} = \frac{1}{2}$

$\boxed{0}\ 0$
$1\ 1$
$0\ 0$
$1\ 1$
$0\ 0$
$1\ 1$
$1\ 1$
$\vdots$

By looking at first entry, I learn everything about the second entry.
$\Rightarrow$ maximally correlated.

~~$I = H(XY)$~~

entropy of join

Random variables X, Y
Joint prob. distribution $p(X, Y)$

$$I = H(X) - H(X|Y) \underset{\text{exercise}}{=} H(X) + H(Y) - H(X,Y)$$

$$H(X|Y) = \sum_y p(y) \sum_x - p(x|y) \log_2 p(x|y)$$

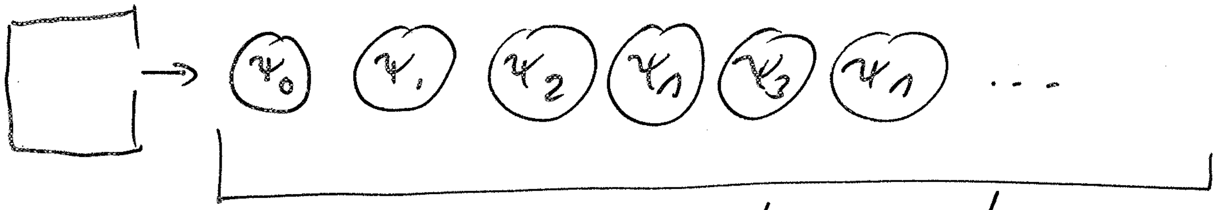
$H(p(x)) = -\frac{1}{2} \log \frac{1}{2} - \frac{1}{2} \log \frac{1}{2} = 1$

$H(p(y)) = \quad 1$

$H(p(x,y)) = 1$

$\Rightarrow I = 1 + 1 - 1 = 1$

Quantum Info & Corr



N two level quantum systems

$\Rightarrow$ Hilbertspace is 2^N dimensional

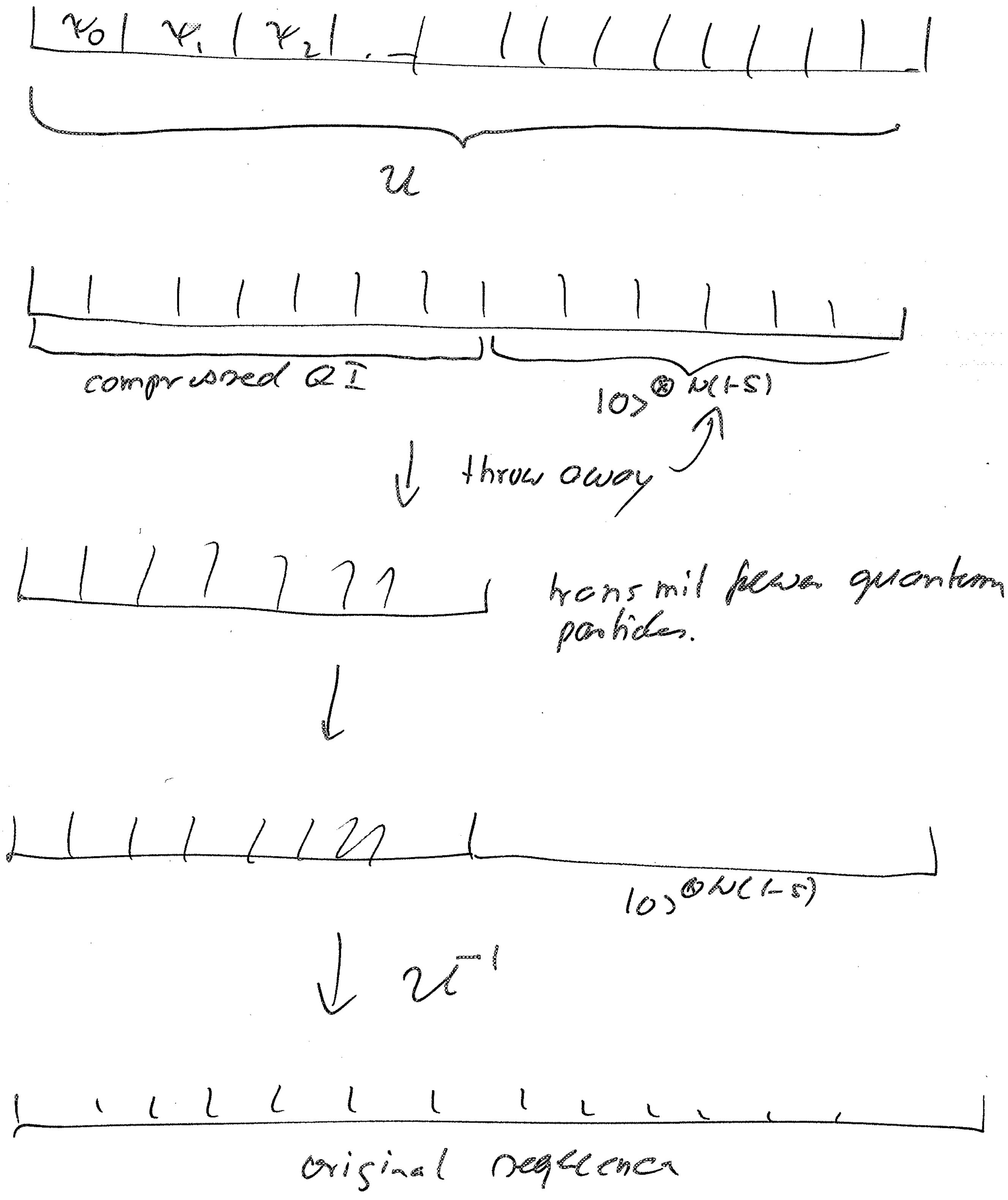
Quantum compression $\rightarrow$ try and represent these quantum state in a smaller Hilbert space.

2^{NS}

$$\rho = \sum_i q_i |\psi_i\rangle\langle\psi_i|$$

$$= \sum_j p_j |\varphi_j\rangle\langle\varphi_j| \qquad \langle\varphi_i|\varphi_j\rangle = \delta_{ij}$$

$\rightsquigarrow$ compression ratio $H(\rho) = \underline{\underline{S(\rho)}} = -\operatorname{tr} \rho \log_2 \rho$


ψ_0 ψ_1 ψ_2 ...
$\mathcal{U}$
compressed QI
$|0\rangle^{\otimes N(1-S)}$
throw away
transmit fewer quantum particles.
$|0\rangle^{\otimes N(1-S)}$
$\mathcal{U}^{-1}$
original sequence

typical sequences of quantum states

$$|\varphi_0\rangle|\varphi_1\rangle|\varphi_1\rangle|\varphi_1\rangle \xrightarrow{U} |\varphi_0\rangle|\varphi_0\rangle|0\rangle|0\rangle$$
$$|\varphi_1\rangle|\varphi_0\rangle|\varphi_1\rangle|\varphi_1\rangle \longrightarrow |\varphi_0\rangle|\varphi_1\rangle|0\rangle|0\rangle$$
$$|\varphi_1\rangle|\varphi_1\rangle|\varphi_0\rangle|\varphi_1\rangle \longrightarrow |\varphi_1\rangle|\varphi_0\rangle|0\rangle|0\rangle$$
$$|\varphi_1\rangle|\varphi_1\rangle|\varphi_1\rangle|\varphi_0\rangle \longrightarrow |\varphi_1\rangle|\varphi_1\rangle|0\rangle|0\rangle$$

Schumacher 1995

Qubit

$$\rho = \tfrac{1}{2}\mathbb{1} \Rightarrow S(\rho) = 1$$

Classical Info   Shannon 1948

0001		00 00	
0010		01 00	
0100	$\xrightarrow{\text{enumerate}}$	10 00	
1000		11 00	

$p_0 = 3/4$

$p_1 = 1/4$

$\downarrow$ compression by a factor $H = -\sum_{i=0}^{1} p_i \log_2 p_i$

total information content of a seq. of N digits is $H \cdot N$

$\hat{=}$ Shannon entropy

Correlations

$X \quad Y$

measure X how much information do you gain about Y

$$
\begin{aligned}
I &= \cancel{H(X) - H(X} \\
&= H(Y) - H(Y|X) \\
&= H(Y) + H(X) - H(X,Y)
\end{aligned}
$$

Quantum case

Schumacher 1993/95

$|\varphi_0\rangle |\varphi_1\rangle |\varphi_0\rangle \ldots$

2-level systems Hilbertspace of N systems has 2^N dimensions.

compress quantum into smaller Hilbert space with dim $2^{NS(\rho)}$

where $\rho = \sum_i p_i |\varphi_i\rangle\langle\varphi_i|$

$S(\rho) = -\operatorname{tr} \rho \log_2 \rho$ von Neumann entropy

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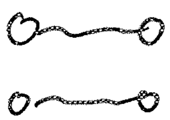
Quantum State Teleportation

Alice is given an unknown quantum state of a qubit

$\rho = \frac{1}{2} \cdot \mathbb{1}$

If we can only do quantum operations locally in Ulm (Alices lab) and in Strathclyde (Bob's lab) & ~~co~~ they can telephone to correlate their actions.



$$(\alpha|00\rangle+\beta|11\rangle)^{\otimes 2} = (\alpha|0_A\rangle|0\rangle_B + \beta|1_A\rangle|1\rangle_B)\otimes(\alpha|0\rangle_A|0\rangle_B + \beta|1\rangle_A|1\rangle_B)$$

$$= \alpha^2|0_A0_A\rangle|00\rangle_B$$
$$+\alpha\beta|01\rangle_A|01\rangle_B$$
$$+\alpha\beta|10\rangle_A|10\rangle_B$$
$$+\beta^2|11\rangle_A|11\rangle_B$$

Alice measures how many of her particles are in state $|1\rangle$.

$$A = 0\cdot|00\rangle\langle 00| + 1\cdot(|01\rangle\langle 01| + |10\rangle\langle 10|) + 2|11\rangle\langle 11|$$

If Alice find eigenvalue $1 \Rightarrow$

$$U\otimes U \quad \frac{|01\rangle|01\rangle + |10\rangle|10\rangle}{\sqrt{2}}$$ in state after measurement

Alice and Bob change their basis locally

$$|00\rangle \to |00\rangle$$
$$|01\rangle \to |01\rangle$$
$$|10\rangle \to |11\rangle$$
$$|11\rangle \to |10\rangle$$

$$U = |00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11|$$
$$UU^\dagger = U^\dagger U = \mathbb{1}$$

$$\frac{|01\rangle|01\rangle + |11\rangle|11\rangle}{\sqrt{2}} = \frac{|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B}{\sqrt{2}}\,|1\rangle_A|1\rangle_B$$

$(\alpha|00\rangle+\beta|11\rangle)^{\otimes N}$

$\Rightarrow$ transform this with unit probability ($N\to\infty$) to $N\cdot H(\alpha^2)$ perfectly entangled pairs.

$\Rightarrow$ The state $\underline{\alpha|00\rangle+\beta|11\rangle}$ contain $H(\alpha^2,\beta^2)$ ebits.

$$\rho_A = \alpha^2|0\rangle\langle 0| + \beta|1\rangle\langle 1|$$

$$H(\alpha^2,\beta^2) = S(\rho_A)$$

$$p_0=\alpha^2$$
$$p_1=\beta^2$$

For mixed states I could define the amount of entanglement as that required to transfer 1 qubit of quantum information.

$\Leftrightarrow$ How many $|\phi^+\rangle\langle\phi^+|$ can I get from $\rho^{\otimes N}$?

$\rho \to |\phi^+\rangle$

=

How many $|\phi^+\rangle\langle\phi^+|$ do I need to make $\rho^{\otimes N}$ using local operations and classical correlations?

$|\phi^+\rangle \to \rho$

Both are valid ways to quantify correlations but they lead to different answers.

Alice & Bob need additional resource.

Alice Bob

Ψ

unknown quantum state

$$|\phi^\pm\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}$$

$$|\psi^\pm\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}$$

$$\underbrace{(\alpha|0\rangle_A + \beta|1\rangle_A)}_{|\psi\rangle} \otimes \frac{|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B}{\sqrt{2}}$$

$$= \alpha|00\rangle_A|0\rangle_B + \alpha|01\rangle_A|1\rangle_B + \beta|10\rangle_A|0\rangle_B + \beta|11\rangle_A|1\rangle_B$$

$$= \alpha(|\phi^+\rangle + |\phi^-\rangle)|0\rangle_B + \alpha(|\psi^+\rangle + |\psi^-\rangle)|1\rangle_B + \beta(|\psi^+\rangle - |\psi^-\rangle)|0\rangle_B + \beta(|\phi^+\rangle - |\phi^-\rangle)|1\rangle_B$$

$$= |\phi^+\rangle_A(\alpha|0\rangle_B + \beta|1\rangle_B)$$
$$+ |\phi^-\rangle_A(\alpha|0\rangle_B - \beta|1\rangle_B)$$
$$+ |\psi^+\rangle(\alpha|1\rangle_B + \beta|0\rangle_B)$$
$$+ |\psi^-\rangle(\alpha|1\rangle_B - \beta|0\rangle_B)$$

$$\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|$$

$$= |\phi^+\rangle_A(\alpha|0\rangle_B + \beta|1\rangle_B)$$
$$+ |\phi^-\rangle_A \sigma_z(\alpha|0\rangle_B + \beta|1\rangle_B)$$
$$+ |\psi^+\rangle_A \sigma_x(\alpha|0\rangle_B + \beta|1\rangle_B)$$
$$+ |\psi^-\rangle_A \sigma_z\sigma_x(\alpha|0\rangle_B + \beta|1\rangle_B)$$

$$A = 1\cdot|\phi^+\rangle\langle\phi^+| + 2|\phi^-\rangle\langle\phi^-| + 3|\psi^+\rangle\langle\psi^+| + 4|\psi^-\rangle\langle\psi^-|$$

Alice makes measurement of A <u>and</u>
she communicates the outcome to Bob.

<u>Ex</u>: Alice finds "2" $\rightsquigarrow$ projects onto $|\phi^-\rangle$

$\rightsquigarrow$ Bobs particle is in state

$$|\phi^-\rangle_A \otimes \sigma_z(\alpha|0\rangle_B + \beta|1\rangle_B)$$

Bob applies σ_z to his particle

$$|\phi^-\rangle \otimes \sigma_z(\sigma_z(\alpha|0\rangle + \beta|1\rangle))$$

$$= |\phi^-\rangle \otimes (\alpha|0\rangle + \beta|1\rangle)$$

$$\sigma_x = \sigma_x^+ \qquad \sigma_x\sigma_x^+ = \sigma_x\sigma_x = \mathbb{1}$$

$$\sigma_x^+\sigma_x = \mathbb{1}$$

<u>Deterministic process</u>

Bennett et al PRL 1993

Transmits quantum information

$\square$ ϕ_0 ϕ ϕ_1 $|\phi_0\rangle$ ...

$\rho = \frac{1}{2}\mathbb{1}$ "unknown state"

$S(\rho) = 1$

1 unit of quantum correlation (1 ebit) are neccessary and sufficient to teleport one qubit of quantum information.

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}} = 1 \text{ ebit}$$

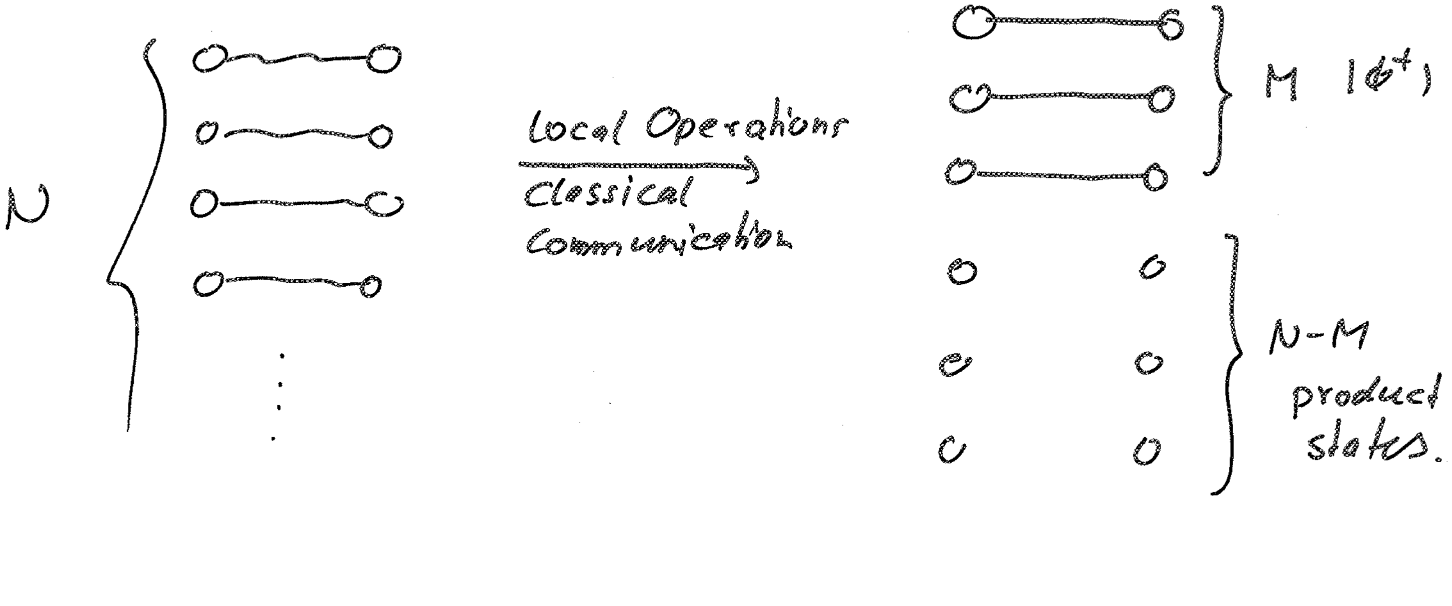
<u>Question</u>: How ~~many~~ much quantum correlations are contained in

$$\alpha|00\rangle + \beta|11\rangle \; ?$$

$$\alpha = \beta = \tfrac{1}{\sqrt{2}} \Rightarrow 1 \text{ ebit}$$

$$\alpha = 0,\ \beta = 1 \Rightarrow |1\rangle|1\rangle \Rightarrow 0 \text{ ebit}$$

$$\left(\alpha|00\rangle_{AB} + \beta|11\rangle_{AB}\right)^{\otimes N}$$



If true for all ρ & σ

$$E_1(\rho) > E_1(\sigma)$$

$$E_2(\rho) > E_2(\sigma)$$

$$\Rightarrow E_1 \equiv E_2$$

Classical correlations

$$I = H(X) + H(Y) - H(X,Y)$$

$$I = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$$

$$|\phi^+\rangle\langle\phi^+| = \rho_{AB}$$

$$S(\rho_{AB}) = 0$$

$$S(\rho_A) = 1$$

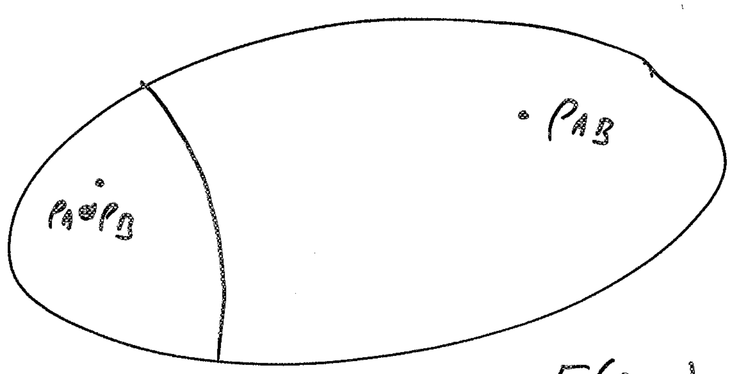
$$S(\rho_B) = 1$$

$$I = 2$$

$$I = S(\rho_{AB} \| \rho_A \otimes \rho_B)$$

$$S(\sigma\|\rho) = \mathrm{tr}(\sigma\log\sigma - \sigma\log\rho)$$

relative entropy



$$E(\rho_{AB}) = \min_{\sigma\in\sum_i p_i\sigma_i^A\otimes\sigma_i^B} S(\rho_{AB}\|\sigma)$$

$$\sum_{i=1} p_i\, \rho_i^A \otimes \rho_i^B$$

Plenio & Virmani, Quant. Inf. Comp. 7, 1 (2007)
~~arXiv: 05~~

~~Vedr.~~ Plenio & Vedral, Cont. Phys. 39, 431 (1998)

Horodecki$^{\otimes 4}$, Rev Mod Phys. ... (2009)