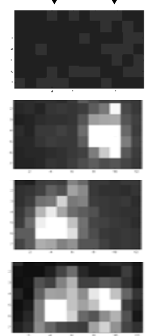
Here are some extensions and problems regarding topics of the two previous lectures:

Measuring observables

Measuring one and two-qubit observables

Results of measuring 1000 copies of a two-qubit state $|\Psi\rangle$

ion 1 ion 2



Detected state	How often observed?
$\lvert\downarrow\downarrow\rangle$	500
$\lvert\uparrow\downarrow\rangle$	100
$\lvert\downarrow\uparrow\rangle$	0
$\lvert\downarrow\downarrow\rangle$	400

Calculate the expectation value of the observables $\sigma_z^{(1)}, \sigma_z^{(2)}, \sigma_z^{(1)}\sigma_z^{(2)}$.

We apply the single-qubit gate $U = e^{i\frac{\pi}{4}\sigma_y^{(1)}} = \frac{1}{\sqrt{2}}(I + i\sigma_y^{(1)})$ to the state $|\Psi\rangle$

before carrying out the fluorescence detection.

Which observables A can we measure in this way?

(with expectation values $\langle\Psi|A|\Psi\rangle$)

Question 1: Expectation value of observables?

For an observable $\mathcal{O}$ having eigenvalues λ_j and eigenvectors $|\phi_j\rangle$, the expectation value of the state ρ is given by $\langle\mathcal{O}\rangle = \sum \lambda_j \langle\phi_j|\rho|\phi_j\rangle$.

For $\sigma_z^{(1)}$, we therefore have

$$\begin{aligned}\langle\sigma_z^{(1)}\rangle &= \frac{1}{N}(N_{\uparrow\uparrow} + N_{\uparrow\downarrow} - N_{\downarrow\uparrow} - N_{\downarrow\downarrow}) \\ &= \tfrac{1}{1000}(500 + 100 - 0 - 400) = 0.2\end{aligned}$$

For $\sigma_z^{(2)}$, we obtain

$$\langle\sigma_z^{(2)}\rangle = \frac{1}{N}(N_{\uparrow\uparrow} - N_{\uparrow\downarrow} + N_{\downarrow\uparrow} - N_{\downarrow\downarrow}) = 0$$

and similarly

$$\langle\sigma_z^{(1)}\sigma_z^{(2)}\rangle = \frac{1}{N}(N_{\uparrow\uparrow} - N_{\uparrow\downarrow} - N_{\downarrow\uparrow} + N_{\downarrow\downarrow}) = 0.8$$

Question 2: Which observables A,B,C are measured?

The fluorescence measurement of ion 1 after applying the unitary U to the state $|\Psi\rangle$ measures the expectation value $\langle U\Psi|\sigma_z^{(1)}|U\Psi\rangle = \langle\Psi|U^\dagger\sigma_z^{(1)}U|\Psi\rangle$

Therefore, we measure the observable

$$A = U^\dagger\sigma_z^{(1)}U = \frac{1}{2}(I - i\sigma_y^{(1)})\sigma_z^{(1)}(I + i\sigma_y^{(1)})$$
$$= \frac{1}{2}(\sigma_z^{(1)} + \sigma_y^{(1)}\underbrace{\sigma_z^{(1)}\sigma_y^{(1)}}_{-\sigma_z^{(1)}} + i\underbrace{[\sigma_z^{(1)}, \sigma_y^{(1)}]}_{-i\sigma_x^{(1)}}) = \sigma_x^{(1)}$$

Similarly, the observable $\sigma_x^{(1)}\sigma_z^{(2)}$ is transformed into

$$B = \sigma_x^{(1)}\sigma_z^{(2)}$$

by the unitary operation whereas

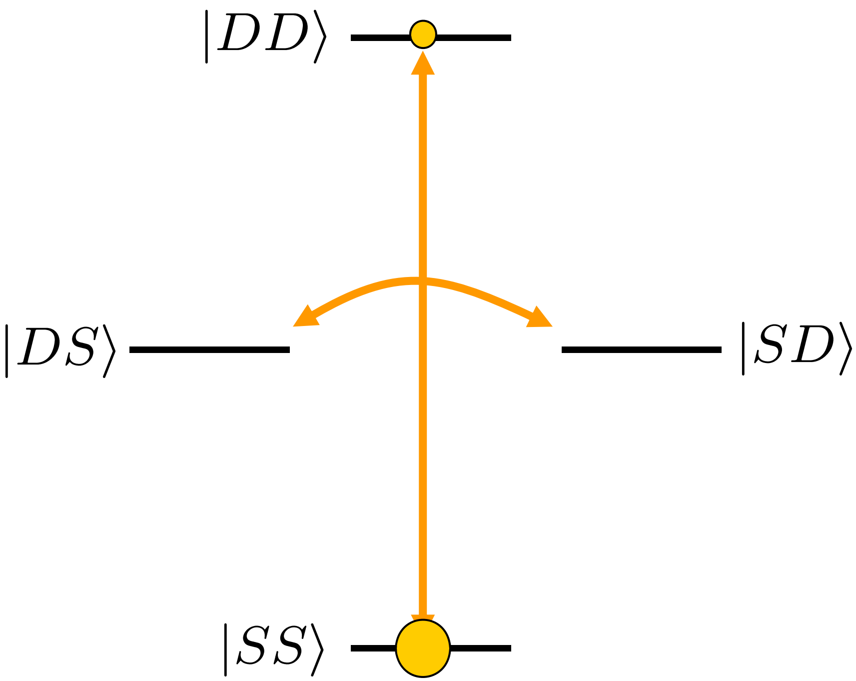
$$C = \sigma_z^{(2)}$$

is not affected the the unitary operating on qubit 1.

Mølmer-Sørensen gate (I) :

Correlated spin flips

Entangling ions by correlated spin flips



Gate action: correlated spin flips

$\boxed{|DS\rangle \leftrightarrow |SD\rangle}$ $\boxed{|SS\rangle \leftrightarrow |DD\rangle}$

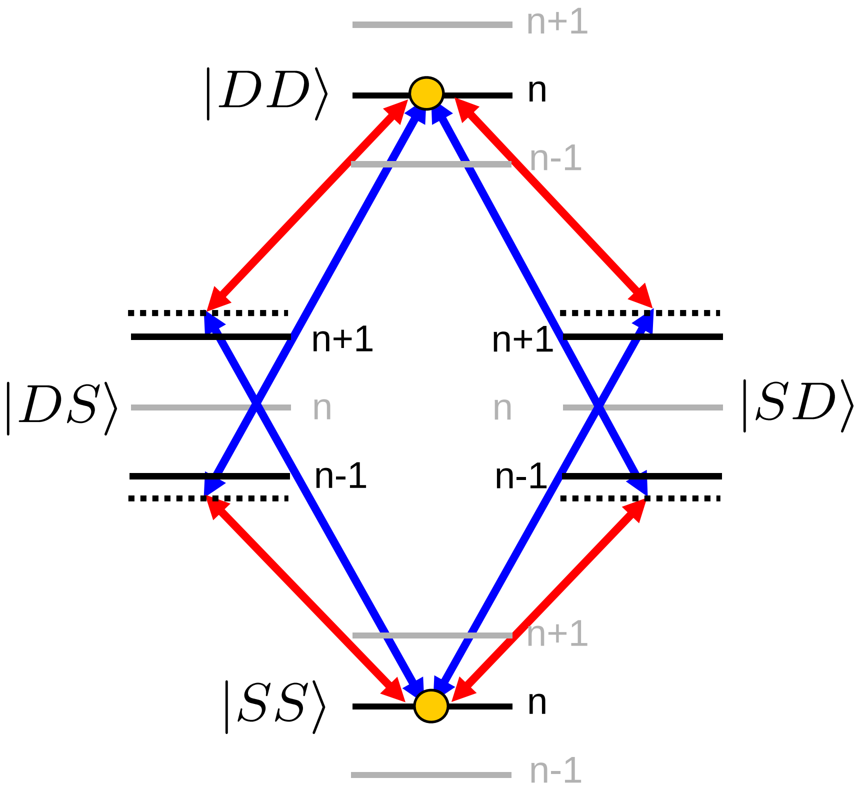
Entanglement generation:

$$|SS\rangle \longrightarrow |SS\rangle + |DD\rangle$$

How do we create correlated spin flips ?

$\longrightarrow$ Couple the ions via the vibrational mode !

Coupling to motional states: Two-photon transition



Gate action: correlated spin flips

$|DS\rangle \leftrightarrow |SD\rangle$ $\quad$ $\boxed{|SS\rangle \leftrightarrow |DD\rangle}$

Bichromatic laser field coupling to

upper motional sideband

lower motional sideband

$$\boxed{H_{eff} = J\sigma_y \otimes \sigma_y}$$

Theory:

A. Sørensen, K. Mølmer, Phys. Rev. Lett. **82**, 1971 (1999)

A. Sørensen, K. Mølmer, Phys. Rev. A **62**, 022311 (2000)

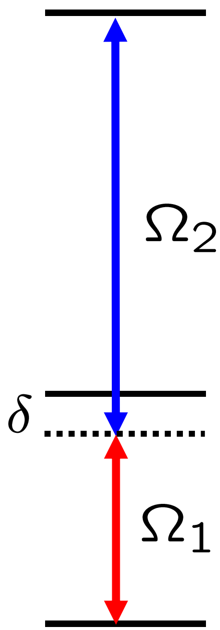
Experiments (Boulder + Ann Arbor):

C. A. Sackett et al, Nature **404**, 256 (2000)

P. Haljan et al., Phys. Rev. A **72**, 062316 (2005)

Two-photon transitions and Molmer-Sorensen gate

Two-photon transition



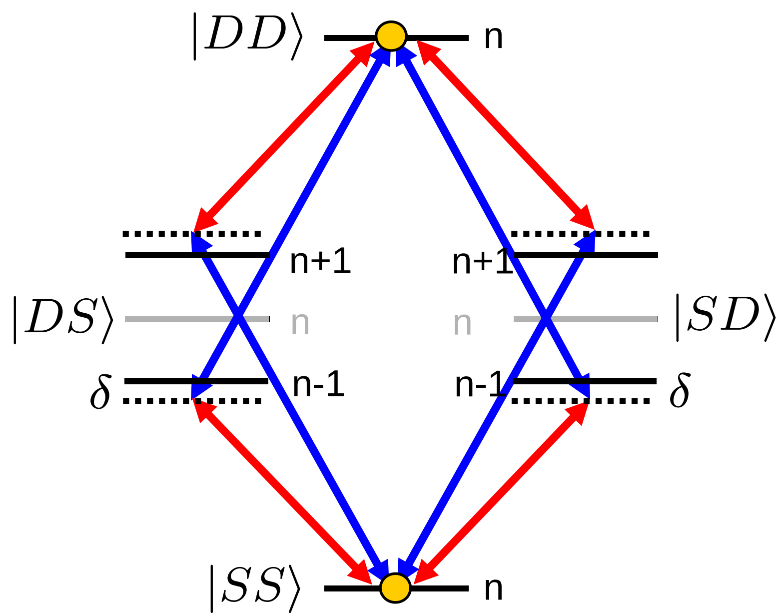
$$\delta \gg \Omega_1, \Omega_2$$

Two-photon transition rate:

$$\Omega_{two-photon} = \frac{\Omega_1 \Omega_2}{\delta}$$

MS gate

Transition	Coupling strength
$\vert S, n\rangle \leftrightarrow \vert D, n+1\rangle$	$\eta\Omega\sqrt{n+1}$
$\vert S, n\rangle \leftrightarrow \vert D, n-1\rangle$	$\eta\Omega\sqrt{n}$



Calculate the two-photon transition rate
in order to show that it
does not depend on the motional state

Show that the two-photon transition rate does not depend on the motional state.

In order to calculate the two-photon coupling strength $\Omega_{SS,DD}$ between the levels $|SS,n\rangle$ and $|DD,n\rangle$, we need to sum up four contributions where the coupling is mediated by the intermediate states $|SD,n+1\rangle$, $|DS,n+1\rangle$, $|SD,n-1\rangle$, $|DS,n-1\rangle$. We find

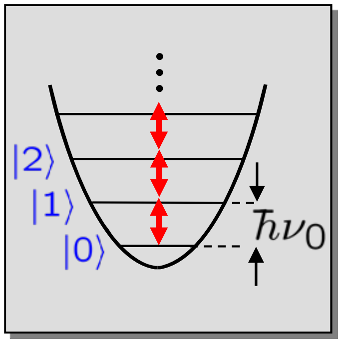
$$\begin{aligned}\Omega_{SS,DD} &= 2\left[\frac{(\eta\Omega\sqrt{n})(\eta\Omega\sqrt{n})}{\delta} + \frac{(\eta\Omega\sqrt{n+1})(\eta\Omega\sqrt{n+1})}{-\delta}\right] \\ &= -2\frac{\eta^2\Omega^2}{\delta}\end{aligned}$$

where we took into consideration that the detuning of the lasers from the intermediate levels is opposite when coupling to levels with $n+1$ and $n-1$ phonons and that the coupling strength on the second step of the transition needs to be calculated for a vibrational phonon number of $n+1$ and $n-1$ respectively.

Driven quantum harmonic oscillator

Driven quantum harmonic oscillator

Harmonic oscillator



$$H(t) = \hbar\nu_0 a^\dagger a \;+ \hbar\Omega i(a^\dagger e^{i\nu t} - a e^{-i\nu t})$$

Drive frequency ν

Interaction picture:

$$H_{int} = \hbar\Omega i(a^\dagger e^{i\delta t} - a e^{-i\delta t})\,, \quad \delta = \nu - \nu_0$$

If the harmonic oscillator is initially in the ground state,

what is ist state after the time

$$\tau = \frac{2\pi}{\delta} \quad ?$$

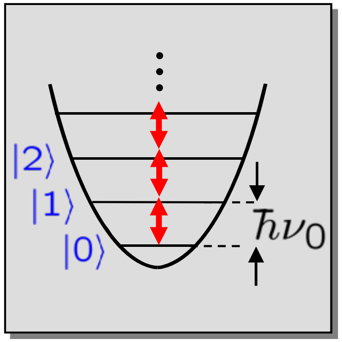
… it has returned to its initial state

$$\psi(t=0) = |0\rangle \longrightarrow \psi(\tau) = |0\rangle \;\cdot e^{i\Phi}$$

but is multiplied by a phase factor.

Driven quantum harmonic oscillator

Harmonic oscillator



$$H(t) = \hbar\nu_0 a^\dagger a \; + \hbar\Omega i(a^\dagger e^{i\nu t} - a e^{-i\nu t})$$

Drive frequency ν

Interaction picture:

$$H_{int} = \hbar\Omega i(a^\dagger e^{i\delta t} - a e^{-i\delta t}) \,, \quad \delta = \nu - \nu_0$$

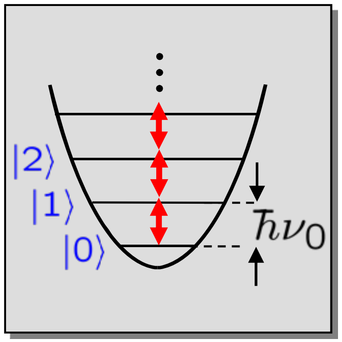
Time evolution:

$$U(t) = \lim_{n\to\infty} \prod_{k=1}^{n} \exp(-\frac{i}{\hbar} H(t_k)\Delta t) \qquad \Delta t = t/n$$

$$t_k = k\Delta t$$

Driven quantum harmonic oscillator

Harmonic oscillator



$$H(t) = \hbar\nu_0 a^\dagger a \; + \hbar\Omega i(a^\dagger e^{i\nu t} - a e^{-i\nu t})$$

Drive frequency ν

Interaction picture:

$$H_{int} = \hbar\Omega i(a^\dagger e^{i\delta t} - a e^{-i\delta t}) \, , \quad \delta = \nu - \nu_0$$

Time evolution:

$$U(t) = \lim_{n\to\infty} \prod_{k=1}^{n} \exp(-\frac{i}{\hbar} H(t_k)\Delta t) \qquad \Delta t = t/n$$

$$t_k = k\Delta t$$

Displacement operator:

$$\hat{D}(\gamma) = e^{\gamma a^\dagger - \gamma^* a}$$

Baker-Campbell-Hausdorff formula:

$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]} \text{ if } [A,[A,B]] = [B,[A,B]] = 0$$

Driven quantum harmonic oscillator

Time evolution:

$$U(t) = \lim_{n\to\infty} \prod_{k=1}^{n} \exp(-\frac{i}{\hbar} H(t_k)\Delta t) = \lim_{n\to\infty} \prod_{k=1}^{n} \hat{D}(\Omega e^{i\delta t_k}\Delta t)$$

$$= \hat{D}(\alpha(t)) e^{i\Phi(t)} \quad \text{with} \qquad \alpha(t) = i\left(\frac{\Omega}{\delta}\right)(1 - e^{i\delta t})$$

$$\Phi(t) = \left(\frac{\Omega}{\delta}\right)^2 (\delta t - \sin\delta t)$$

Displacement operator:

$$\hat{D}(\gamma) = e^{\gamma a^\dagger - \gamma^* a}$$

$$\boxed{\hat{D}(\alpha)\hat{D}(\beta) = \hat{D}(\alpha+\beta) e^{i\mathrm{Im}(\alpha\beta^*)}}$$

Baker-Campbell-Hausdorff formula:

$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]} \quad \text{if} \quad [A,[A,B]] = [B,[A,B]] = 0$$

Driven quantum harmonic oscillator

Time evolution:

$$U(t) = \lim_{n\to\infty} \prod_{k=1}^{n} \exp(-\frac{i}{\hbar} H(t_k)\Delta t) = \lim_{n\to\infty} \prod_{k=1}^{n} \hat{D}(\Omega e^{i\delta t_k}\Delta t)$$

$$= \hat{D}(\alpha(t)) e^{i\Phi(t)} \quad \text{with} \qquad \alpha(t) = i\left(\frac{\Omega}{\delta}\right)(1 - e^{i\delta t})$$

$$\Phi(t) = \left(\frac{\Omega}{\delta}\right)^2 (\delta t - \sin \delta t)$$

Time evolution for the ground state: $\psi(t=0) = |0\rangle$

$$U(t)|0\rangle = \hat{D}(\alpha(t)) e^{i\Phi(t)} |0\rangle$$

$$= e^{i\Phi(t)} |\alpha(t)\rangle$$

$|\alpha(t)\rangle$: coherent state

Time evolution in phase space

$$U(t)|0\rangle = e^{i\Phi(t)}|\alpha(t)\rangle$$

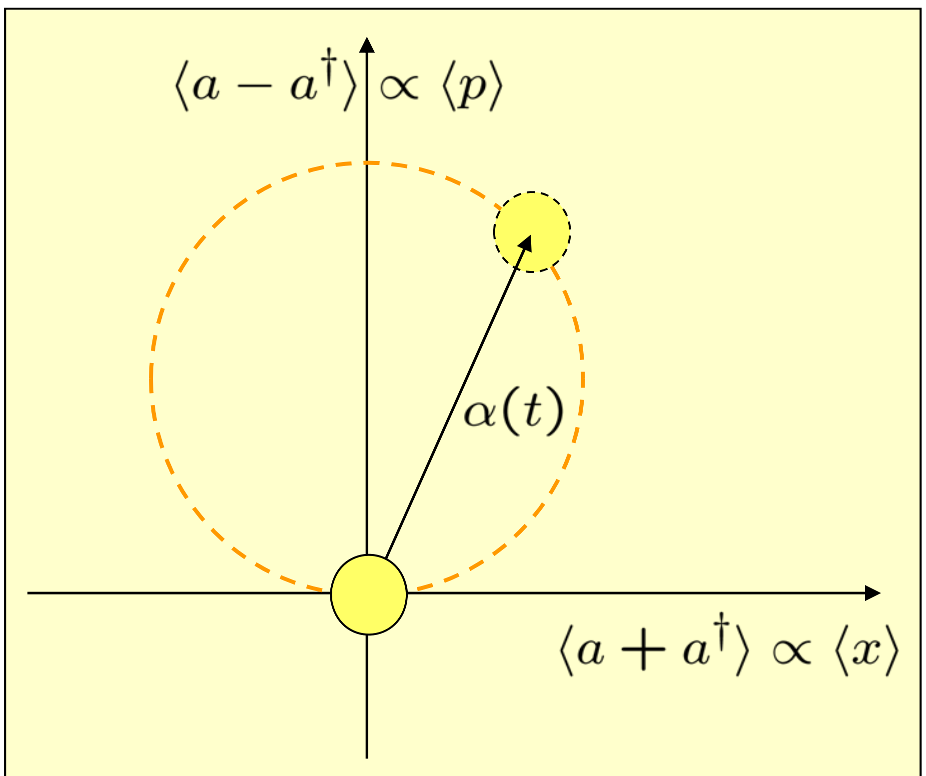
$$\alpha(t) = i\left(\frac{\Omega}{\delta}\right)(1 - e^{i\delta t})$$

$$\Phi(t) = \left(\frac{\Omega}{\delta}\right)^2 (\delta t - \sin\delta t)$$

For $t^* = \dfrac{2\pi}{\delta}$:

$= 0$

$= 2\pi\left(\dfrac{\Omega}{\delta}\right)^2$



$$U(t^*)|0\rangle = e^{i\Phi(t^*)}|0\rangle$$

Mølmer-Sørensen gate (II) :

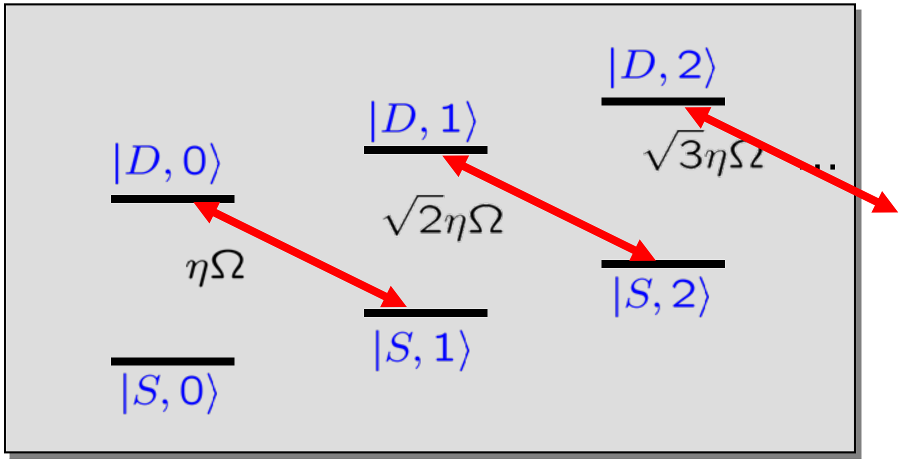
State-dependent
driven quantum harmonic oscillator

Red sideband

Laser-ion interaction: Coupling internal and motional states

$$\omega_{laser} = \omega_0 - \nu$$

Joint energy levels



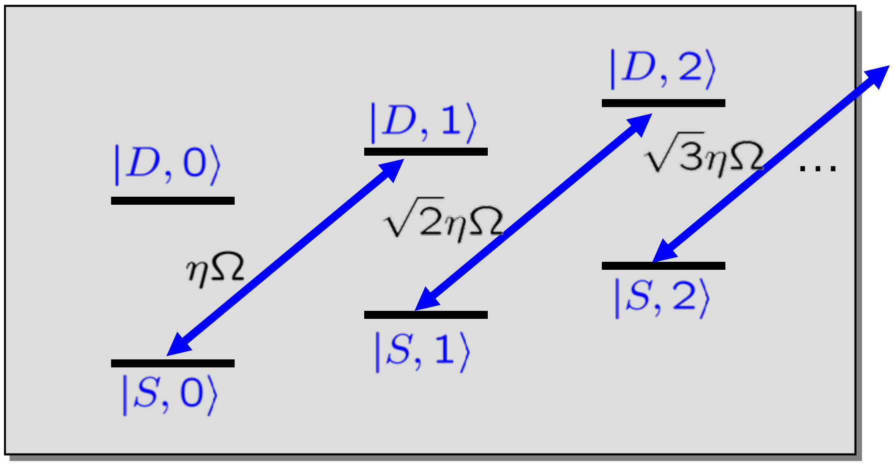
Red sideband

$$H_{red} \propto \eta\Omega(\sigma_- a^\dagger e^{i\phi} + \sigma_+ a e^{-i\phi})$$

Blue sideband

Laser-ion interaction: Coupling internal and motional states

Joint energy levels



$$\omega_{laser} = \omega_0 + \nu$$

Blue sideband

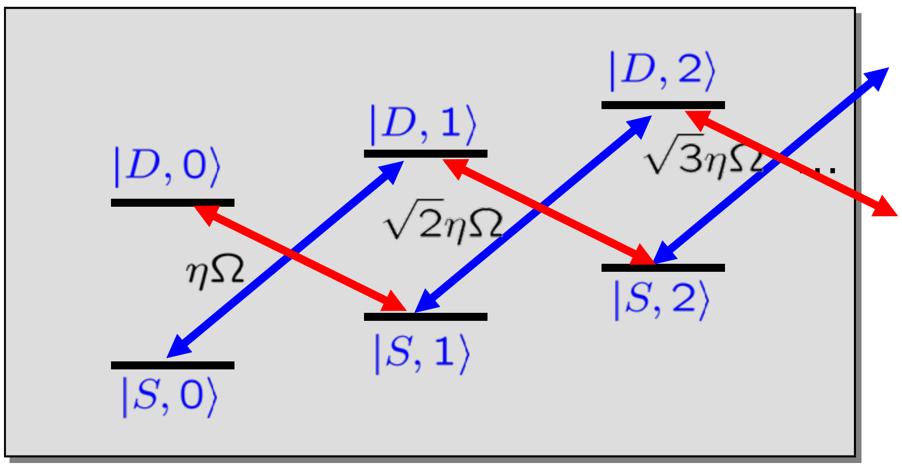
$$H_{blue} \propto \eta\Omega(\sigma_+ a^\dagger e^{i\phi} + \sigma_- a e^{-i\phi})$$

Bichromatic coupling

Laser-ion interaction: Coupling internal and motional states

$$\omega_{laser} = \omega_0 \pm \nu$$

Joint energy levels



Blue + red sideband

$$H_{blue} \propto \eta\Omega(\sigma_+ a^\dagger e^{i\phi} + \sigma_- a e^{-i\phi})$$

$$H_{red} \propto \eta\Omega(\sigma_- a^\dagger e^{i\phi} + \sigma_+ a e^{-i\phi})$$

State-dependent displacement force:

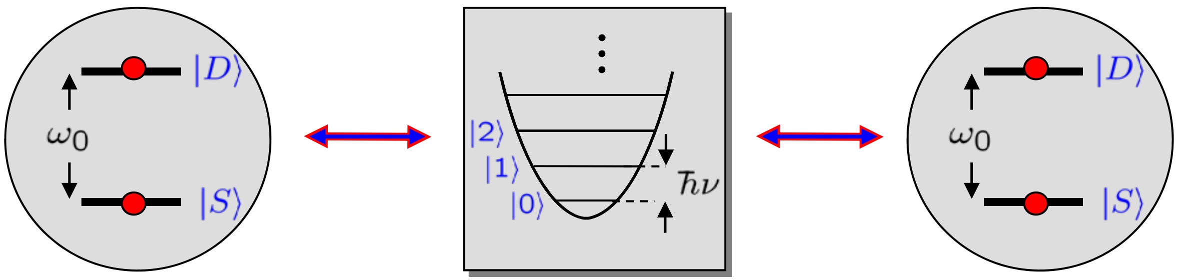
$$H_{bichr} = H_{blue} + H_{red} \propto \eta\Omega(a^\dagger e^{i\phi} - a e^{-i\phi})\sigma_y$$

How do the Hamiltonians change if the lasers are detuned by δ from the transition?

... $\phi \longrightarrow \phi(t) = \delta t$ becomes a time-dependent phase

Mølmer-Sørensen gate

Bichromatic excitation: Two ions with equal couplings



$$H_{bichr} = H_{blue} + H_{red} \propto \eta\Omega(a^\dagger e^{i\phi} - a e^{-i\phi})\,\underbrace{\left(\sigma_y^{(1)} + \sigma_y^{(2)}\right)}_{= S_y}$$

Copropagating bichromatic lasers:

$$\omega_r = \omega_0 - (\nu + \epsilon)$$
$$\omega_b = \omega_0 + (\nu + \epsilon)$$

$$H_{bichr}(t) = i\hbar\eta\Omega(e^{i\epsilon t}a^\dagger - e^{-i\epsilon t}a)\,S_y$$

$$\omega_b + \omega_r = 2\omega_0$$

The Molmer-Sorensen Hamiltonian describes a driven harmonic oscillator with a coupling strength that depends on the internal states of both qubits.

State-dependent driven quantum oscillator

Two ions interacting with a bichromatic laser field

$$H_{bichr}(t) = i\hbar\eta\Omega(e^{i\epsilon t}a^\dagger - e^{-i\epsilon t}a)\, S_y$$

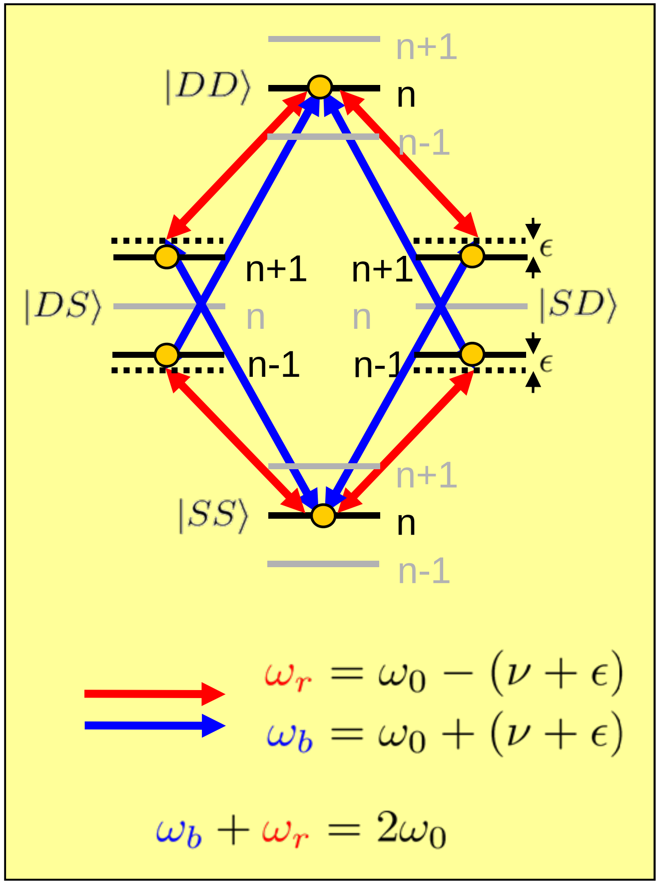
$$[H_{bichr}(t_1), H_{bichr}(t_2)] \propto S_y^2$$

Time evolution:

$$U(t) = \hat{D}(\alpha(t)S_y)\, e^{i\Phi(t)S_y^2}$$

$$\alpha(t) = i\left(\frac{\eta\Omega}{\epsilon}\right)(1 - e^{i\epsilon t})$$

$$\Phi(t) = \left(\frac{\eta\Omega}{\epsilon}\right)^2 (\epsilon t - \sin \epsilon t)$$



For $t^* = \frac{2\pi}{\epsilon}$: $U(t^*) = e^{i\Phi(t^*)S_y^2} \longrightarrow H_{eff} = -\hbar\frac{(\eta\Omega)^2}{\epsilon}S_y^2$

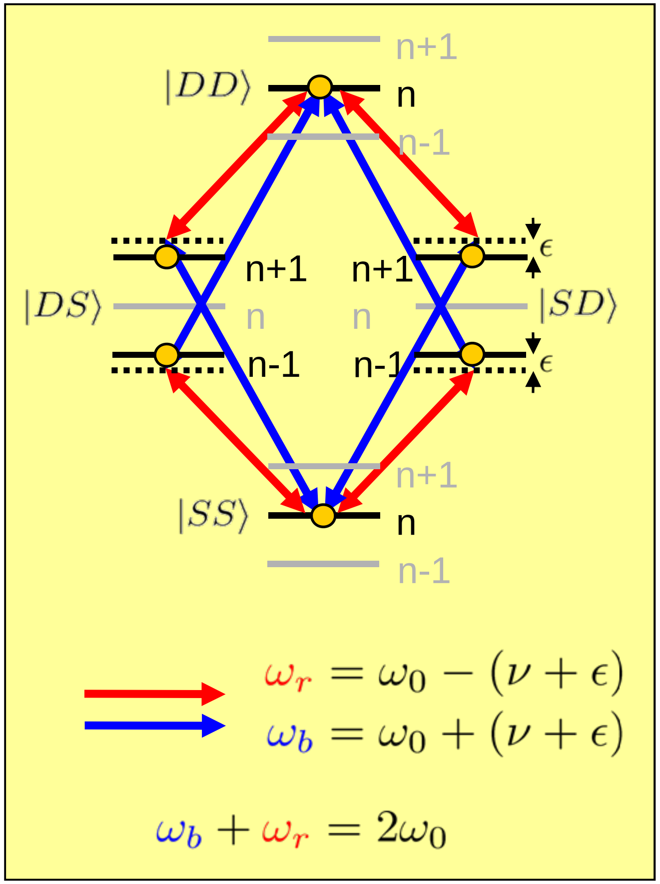
Entangling two ions with the Mølmer-Sørensen gate

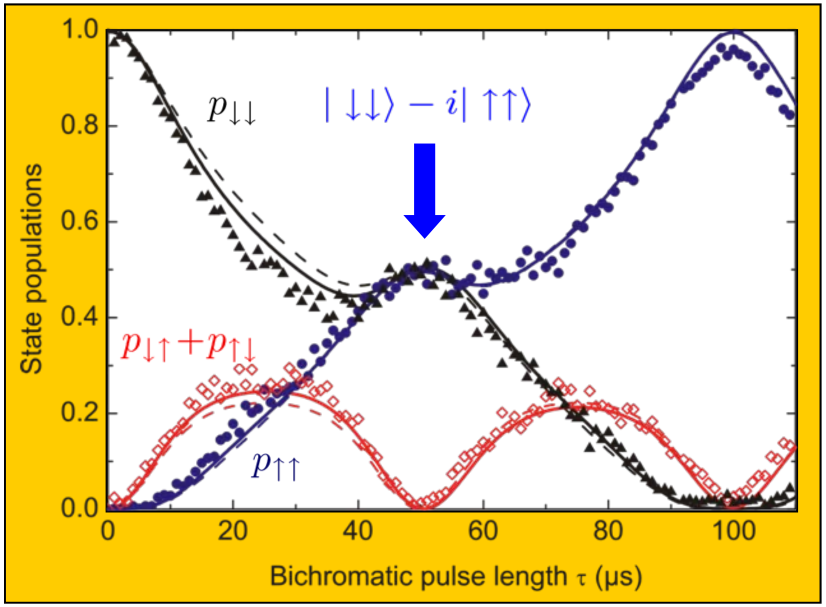
$$S_y^2 = (\sigma_y^{(1)} + \sigma_y^{(2)})^2 = 2(I + \sigma_y^{(1)}\sigma_y^{(2)})$$

For $\Omega = \frac{\epsilon}{4\eta}$:

$$U(t^*) = e^{i\frac{\pi}{4}\sigma_y^{(1)}\sigma_y^{(2)}} = \frac{1}{\sqrt{2}}(I + i\sigma_y^{(1)}\sigma_y^{(2)})$$

$$U(t^*)|\downarrow\downarrow\rangle = \frac{1}{\sqrt{2}}(|\downarrow\downarrow\rangle - i|\uparrow\uparrow\rangle)$$





A. Sørensen, K. Mølmer, Phys. Rev. A **62**, 022311 (2000)

P.J. Lee et al., J. Opt. B **7**, S371 (2005)

C. F. Roos, New. J. Phys. **10**, 013002 (2008)

Conditional phase gate:

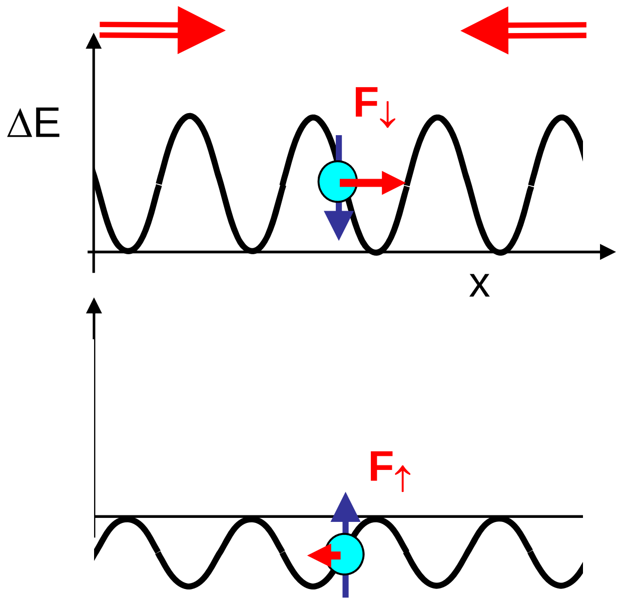
State-dependent
driven quantum harmonic oscillator

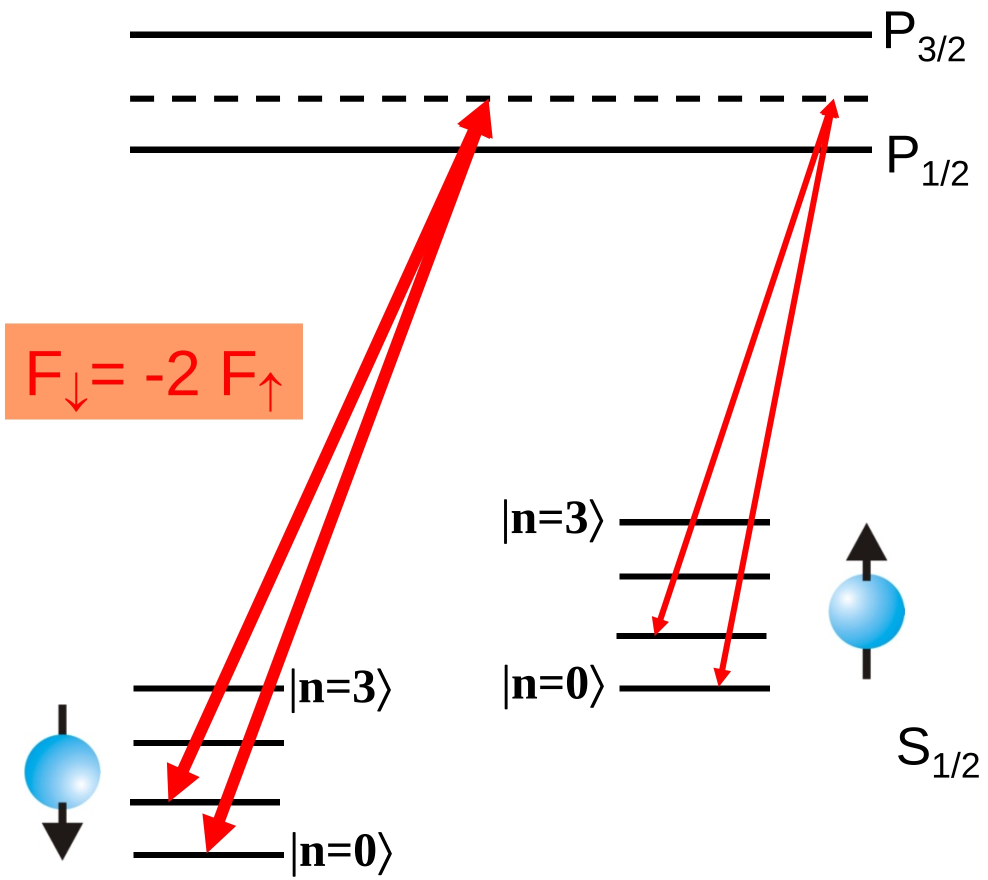
Another entangling gate operation exists that can also be described by a state-dependently driven oscillator. In this gate, the ion motion is off-resonantly excited by a moving standing wave that couples differently to the qubit states of two qubit encoded in hyperfine ground states. While the physical mechanism is different from the Molmer-Sorensen gate, ist mathematical description is the same.

Entangling interactions: controlled phase gate

Use Raman beams that couple the motional states (but not internal states)

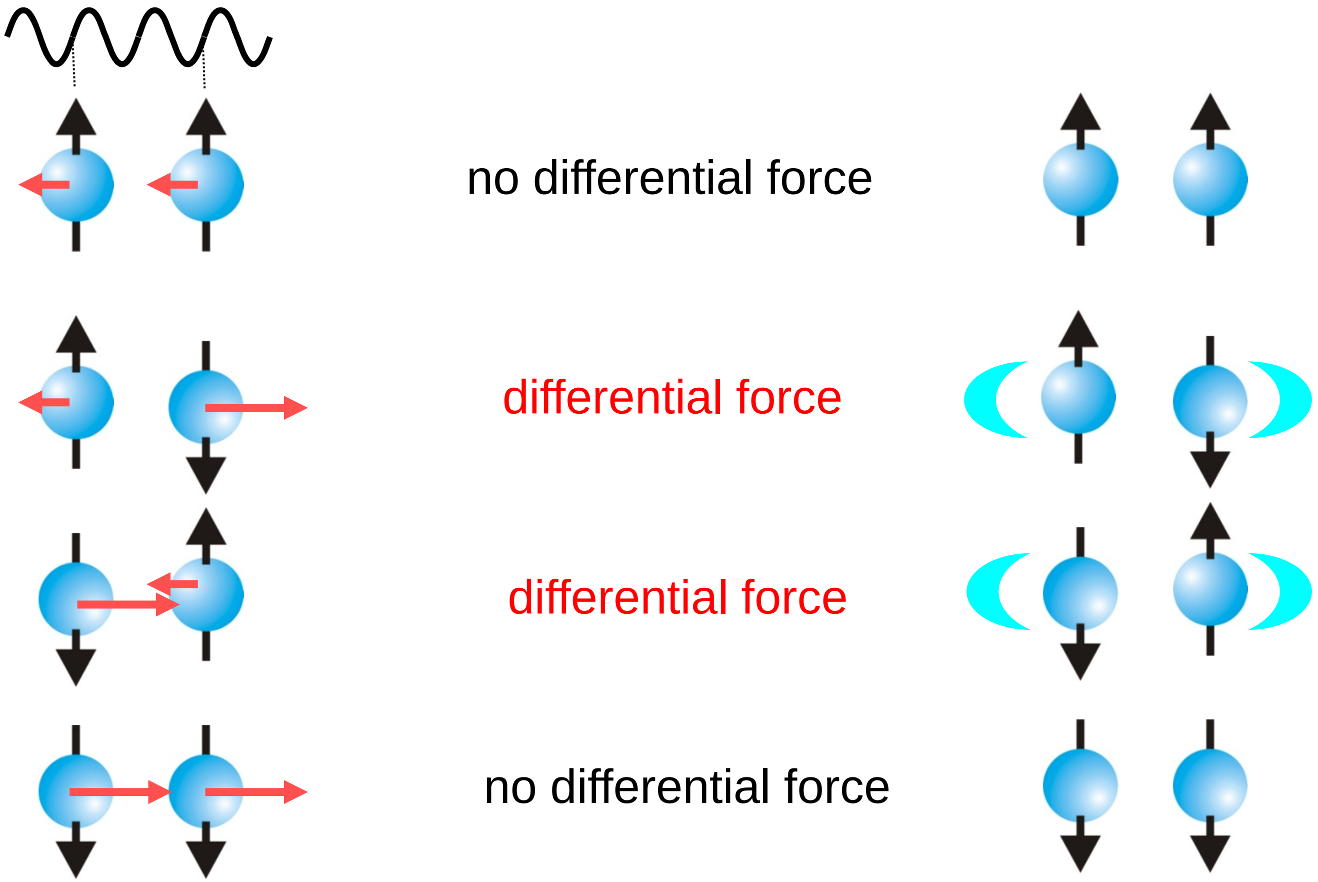
Raman beams form (moving) standing wave: spatial light shifts





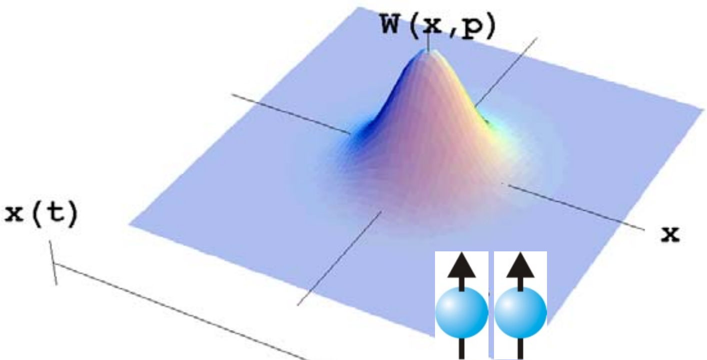
State dependent optical dipole force

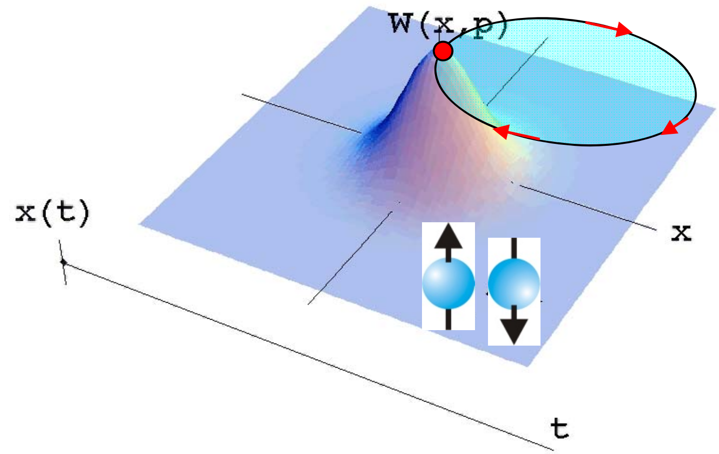
Stretch mode excitation



$$H(t) = (\alpha(t)a + \alpha^*(t)a^\dagger)(\sigma_z^{(1)} - \sigma_z^{(2)})$$

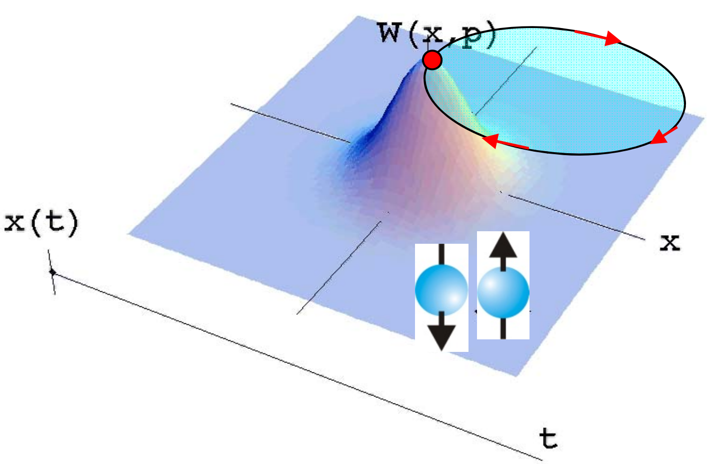
Phase space picture





Boulder group :
Gate fidelity: 97%
Gate time: 7 μs (ca. $25/\nu_{COM}$)

D. Leibfried *et al.*, Nature **422**, 414 (2003)
Theory: Milburn, Fortschr. Phys. **48**, 9(2000)



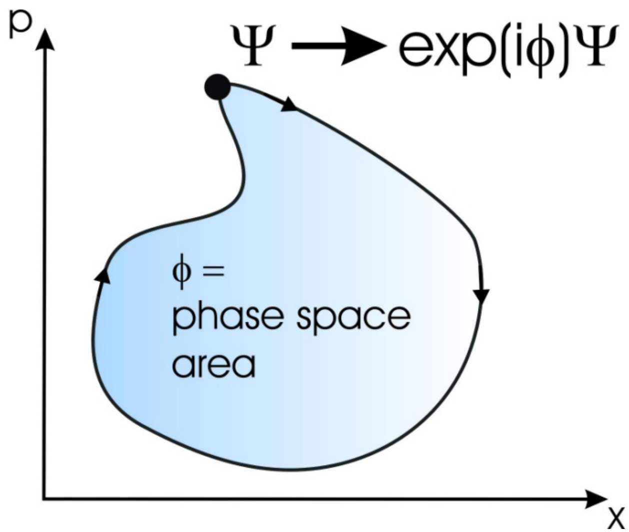
$$|\downarrow\downarrow\rangle \rightarrow |\downarrow\downarrow\rangle$$

$$|\downarrow\uparrow\rangle \rightarrow e^{i\phi}|\downarrow\uparrow\rangle$$

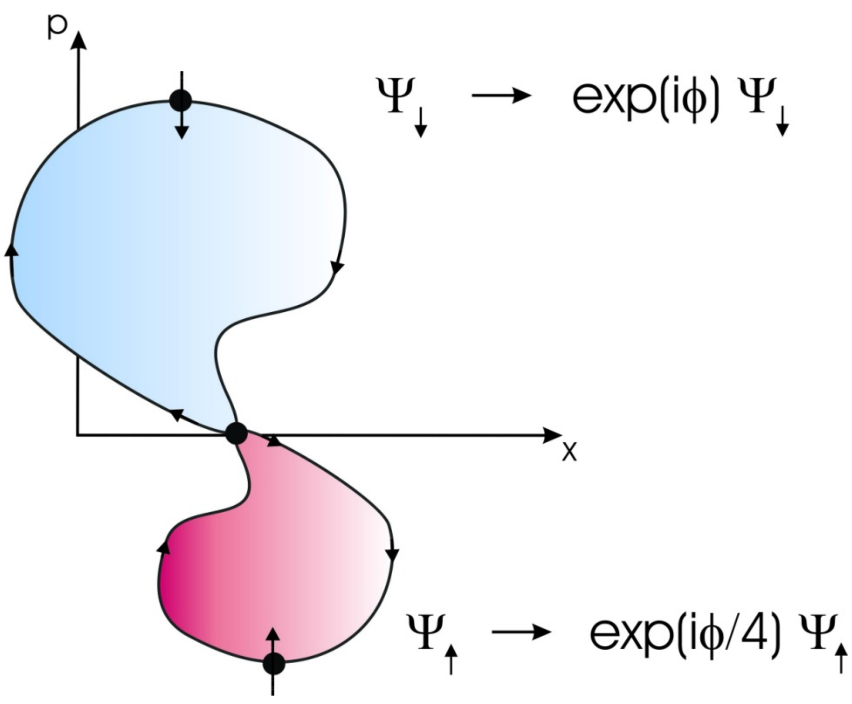
$$|\uparrow\downarrow\rangle \rightarrow e^{i\phi}|\uparrow\downarrow\rangle$$

$$|\uparrow\uparrow\rangle \rightarrow |\uparrow\uparrow\rangle$$

Geometric phase gate



1) coherent displacement along closed path will shift phase of the quantum state, phase independent of details like speed of traversal, etc.



2) the sign and magnitude of coherent displacements can be made internal-state dependent
(see e.g. Science **272**, 1131 (1996))

-no ground state cooling
-no individual addressing
-robust against “small” deformations of the path
-relative phase of successive displacements irrelevant

G. J. Milburn *et al.*, Fortschr. Physik 48, 801 (2000).
X. Wang *et al.*, Phys. Rev. Lett. 86, 3907 (2001).